Separable differential equations

Ordinary differential equations (ODEs)

A separable, first-order differential equation is an equation in the following form

$$y' = f(x)g(y),$$

where $f(x)$ and $g(y)$ are functions of x and y, respectively. The dependent variable is y; the independent variable is x. We can easily integrate functions in this form by separating variables.

$$y' = f(x)g(y)$$

$$\frac{dy}{dx} = f(x)g(y)$$

$$dy = f(x)g(y)\ dx$$

$$\frac{dy}{g(y)} = f(x)\ dx$$

$$\frac{1}{g(y)}\ dy = f(x)\ dx$$

$$\int \frac{1}{g(y)}\ dy = \int f(x)\ dx$$

Sometimes in our final answer, we'll be able to express y explicitly as a function of x, but not always. When we can't, we just have to be satisfied with an implicit function, where y and x are not cleanly separated by the $=$ sign.

Example

Solve the differential equation.

$$y' = y^2 \sin x$$

First, we'll write the equation in Leibniz notation. This makes it easier for us to separate the variables.

$$\frac{dy}{dx} = y^2 \sin x$$

Next, we'll separate the variables, collecting y's on the left and x's on the right.

$$dy = y^2 \sin x \ dx$$

$$\frac{dy}{y^2} = \sin x \ dx$$

$$\frac{1}{y^2} \ dy = \sin x \ dx$$

With variables separated, and integrating both sides, we get

$$\int \frac{1}{y^2} \ dy = \int \sin x \ dx$$

$$\int y^{-2} \ dy = \int \sin x \ dx$$

$$-y^{-1} = -\cos x + C$$

Note: You can leave out the constant of integration on the left side, because in future steps it would be absorbed into the constant on the right side.

$$-\frac{1}{y} = -\cos x + C$$

$$\frac{1}{y} = \cos x + C$$

Note: We just multiplied through both sides by -1, but we didn't change the sign on C, because the negative can always be absorbed into the constant.

$$1 = y(\cos x + C)$$

$$y = \frac{1}{\cos x + C}$$

Sometimes we'll encounter separable differential equations with initial conditions provided. Using the same method we used in the last example, we can find the general solution, and then plug in the initial condition(s) to find a particular solution to the differential equation.