1: Quadratic Functions

Functions

In graphs, there's a relationship between x and y. For example, $y = mx + c$, where x is the input and y is the output. We call this relationship a function.

As such, we may write a quadratic function of x as $f(x) = ax^2 + bx + c$; or we may write a linear function of x as $f(x) = mx + c$, where substituting the value of x in the function gives y.

*When manipulating a function, we cannot $+, -, \times, \div$ on both sides of the equation.

<u>Example Question:</u>

Given that $f(x) = 4x^2 - 3,\ find$

(i) its value when $x = 0$

(ii) all possible real values of $f(x)$

(i) Sub $x = 0$:

$f(0) = 4(0)^2 - 3 = -3$

(ii) Since $x^2 \geq 0$

$4x^2 \geq 0 \Leftrightarrow 4x^2 \geq 4(0)$

$4x^2 - 3 \geq -3$

$f(x) \geq -3$

Completing the Square

<u>Recall:</u> In eMath, we've learnt to convert any $x^2 + bx + c$ into the form $(x - h)^2 + k$.

In aMath, we will learn to convert any $ax^2 + bx + c$ into the form $a(x - h)^2 + k$, where $a \neq 0$.

This can help us find the minimum or maximum value of any $f(x)$, as the completed square form will allow us to easily identify the inequality of $f(x)$.

Example, if $f(x) = x^2 + 6x - 4$

We cannot just deduce that $f(x) \geq 6x - 4$

Even though this is true, this doesn't help us find the true value of the maximum/minimum.

Example (where $a = 1$)

Given that $f(x) = x^2 + 8x - 2$, find

(i) all possible values of $f(x)$

(ii) **Hence,** the minimum value of $f(x)$ and its corresponding value of x.

(i) $f(x) = x^2 + 8x - 2$

$= [(x + 4)^2 - 4^2] - 2$

$= (x + 4)^2 - 16 - 2$

$= (x + 4)^2 - 18$

Since $(x + 4)^2 \geq 0$

$(x + 4)^2 - 18 \geq -18$

$f(x) \geq -18$

(ii) The minimum value of $f(x)$ is -18 when $x = -4$.

Example (where $a \neq 0, 1$): [Factoring out the first coefficient]

Given that $f(x) = -3x^2 + 5x - 4$,

find the minimum or maximum value of $f(x)$ and its corresponding value of x.

$f(x) = -3(x^2 - \frac{5}{3}x) - 4$

$= -3[(x - \frac{5}{6})^2 - (-\frac{5}{6})^2] - 4$

$= -3(x - \frac{5}{6})^2 + 3(-\frac{5}{6})^2 - 4$

$= -3(x - \frac{5}{6})^2 - \frac{23}{12}$

Since $(x - \frac{5}{6})^2 \geq 0$

$-3(x - \frac{5}{6})^2 \leq 0$ (we flip the sign when multiplying by negative number in inequalities)

$-3(x - \frac{5}{6})^2 - \frac{23}{12} \leq -\frac{23}{12}$

$f(x) \leq -\frac{23}{12}$

Hence, the maximum value of $f(x)$ is $-\frac{23}{12}$ when $x = \frac{5}{6}$.

Example [Showing Questions, Another Example on Page 4]

Show that $-4x^2 + 2x + 7$ is always negative.

Let $f(x) = -4x^2 + 2x - 7$

$f(x) = -4(x^2 - \frac{1}{2}x) - 7$

$= -4[(x - \frac{1}{4})^2 - (-\frac{1}{4})^2] - 7$

$= -4(x - \frac{1}{4})^2 + 4(-\frac{1}{4})^2 - 7$

$= -4(x - \frac{1}{4})^2 - \frac{27}{4}$

Since $(x - \frac{1}{4})^2 \geq 0$

$-4(x - \frac{1}{4})^2 \leq 0$ (inequality rules)

$-4(x - \frac{1}{4})^2 - \frac{27}{4} \leq -\frac{27}{4}$

$f(x) \leq -\frac{27}{4} < 0$

As such, $-4x^2 + 2x - 7$ is always negative.

Graphical Representation of Quadratic Functions

To find specific points on the graph, it is not efficient to use the standard form of the equation. As such, we will have to convert the equation/function into other forms like the factorized form or the completed square form.

The turning point or vertex of a graph is the point when the graph changes direction. This point also lies on the line of symmetry.

To note:

$y = ax^2 + bx + c$ is good to find the y-intercept, which is c

Another use is to find whether the graph is ∪-shaped (when $a > 0$) or ∩-shaped (when $a < 0$)

Another use is to find the line of symmetry, defined as $x = \frac{-b}{2a}$ (not in syllabus, Vieta's formula)

$y = a(x - p)(x - q)$ is good to find the x-intercepts, which is p and q

Another way to find the x-intercepts is by using the quadratic formula.

Another use is to find the line of symmetry, defined as $x = \frac{p+q}{2}$

$y = a(x - h)^2 + k$ is good to find the turning point, which is (h, k)

Examples: Find the intercepts (if any) and the coordinates of the turning point:

(a) $y = 2x^2 + 4x - 3$

(b) $y = 3(x - 2)^2 + 7$

(c) $y = 4(4x + 3)(2x - 1)$

(a) $y = 2x^2 + 4x - 3$

y-intercept = -3

$$y = 2(x^2 + 2x) - 3$$
$$= 2[(x + 1)^2 - 1] - 3$$
$$= 2(x + 1)^2 - 2(1) - 3$$
$$= 2(x + 1)^2 - 5$$

turning point = $(-1, -5)$

Sub $y = 0$:

$$2(x + 1)^2 - 5 = 0$$
$$2(x + 1)^2 = 5$$
$$x + 1 = \pm\sqrt{\frac{5}{2}}$$
$$x = -1 \pm \sqrt{\frac{5}{2}}$$
$$= 0.581 \text{ or } -2.58 \ (3\ s.f.)$$

(b) $y = 3(x - 2)^2 + 7$

turning point = $(2, 7)$

Sub $x = 0$:

$$y = 3(0 - 2)^2 + 7$$
$$= 3(-2)^2 + 7$$
$$= 3(4) + 7$$
$$= 19$$

y-intercept = 19

Sub $y = 0$:

$$3(x - 2)^2 + 7 = 0$$
$$x - 2 = \pm\sqrt{\frac{-7}{3}}$$

There is no x-intercepts as there are no real roots to this equation.

(c) $y = 4(4x + 3)(2x - 1)$

Sub $y = 0$:

$$4(4x + 3)(2x - 1) = 0$$
$$4x + 3 = 0 \text{ or } 2x - 1 = 0$$
$$x = \frac{-3}{4} \qquad x = \frac{1}{2}$$

line of symmetry: $x = \frac{-0.75+0.5}{2}$

$$= \frac{-1}{8}$$

Sub $x = \frac{-1}{8}$:

$$y = 4[4(\tfrac{-1}{8}) + 3][2(\tfrac{-1}{8}) - 1]$$
$$= \frac{-25}{2}$$

turning point = $(-\frac{1}{8}, -\frac{25}{2})$

By partial expansion,

y-intercept= $4(3)(-1) = -12$

[Example: Showing Questions] Prove that $2x^2 + 2x > -3$ for all real values of x.

Let $f(x) = 2x^2 + 2x$

$$f(x) = 2x^2 + 2x$$
$$= 2(x^2 + x)$$
$$= 2[(x + \tfrac{1}{2})^2 - (\tfrac{1}{2})^2]$$
$$= 2(x + \tfrac{1}{2})^2 - 2(\tfrac{1}{2})^2$$
$$= 2(x + \tfrac{1}{2})^2 - \tfrac{1}{2}$$

Since $2(x + \frac{1}{2})^2 \geq 0$

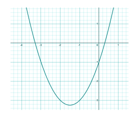
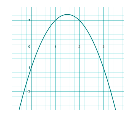
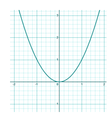
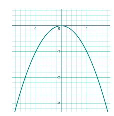
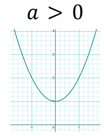
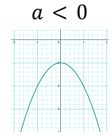
$$2(x + \tfrac{1}{2})^2 - \tfrac{1}{2} \geq -\tfrac{1}{2}$$
$$f(x) \geq -\tfrac{1}{2} > -3$$

Therefore, $2x^2 + 2x > -3$ for all real values of x.

Discriminant, Roots and x-intercepts

We've learnt previously that we can find the roots of an equation using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The expression $b^2 - 4ac$ is called the discriminant, which can be used to find how many x-intercepts are there in any quadratic equation $ax^2 + bx + c = 0$. Hence, it can also be used to find how many (real) roots are there in the equation.

Discriminant $(b^2 - 4ac)$	Nature of roots of $ax^2 + bx + c = 0$	Number of x-intercepts	Graph of $y = ax^2 + bx + c$
> 0	2 real and distinct roots	2	$a > 0$   $a < 0$ The curve intersects the x-axis at 2 points
$= 0$	2 real and equal (repeated) roots	1	$a > 0$   $a < 0$ The curve intersects the x-axis at 1 point (or touches the x-axis)
< 0	No real roots	0	$a > 0$  $a < 0$  The curve does not intersect the x-axis

*If the discriminant < 0 and $a > 0$, the graph lies entirely above the x-axis.

*If the discriminant < 0 and $a < 0$, the graph lies entirely below the x-axis.

Examples: Determine the number of x-intercepts for

(i) $y = x^2 + 3x - 5$

(ii) $y = (2 - x)^2 + 1 \Leftrightarrow y = x^2 - 4x + 5$

(i) Discriminant $= (3)^2 - 4(1)(-5) = 29 > 0$

The graph of $y = x^2 + 3x - 5$ has 2 x-intercepts.

(ii) Discriminant $= (-4)^2 - 4(1)(5) = -4 < 0$

The graph of $y = (2 - x)^2 + 1$ has 0 x-intercepts.

Prove that $y = mx^2 - 4x + m$ touches the x-axis if $m = \pm 2$.

Discriminant = $(-4)^2 - 4(m)(m)$

$= 16 - 4m^2$

Since $m = \pm 2$, $m^2 = 4$

$16 - 4m^2 = 16 - 4(4)$

$= 0$

Therefore, $y = mx^2 - 4x + m$ touches the x-axis when $m = \pm 2$.

Explain why the graph $y = x^2 - 2x + 2 - p$, where $p > 1$, intersects the x-axis at 2 points.

Discriminant = $(-2)^2 - 4(1)(2 - p)$

$= 4 - 8 + 4p$

$= 4p - 4$

Since $p > 1$,

$4p > 4$

$4p - 4 > 0$

$\Delta > 0$ (to refer to the discriminant you may use D or Δ)

Therefore, $y = x^2 - 2x + 2 - p$, where $p > 1$, intersects the x-axis at 2 points.

Find the range of values of k for which

(i) $3x^2 + x + k$ is always positive, **(ii)** the graph of $y = kx^2 + x + 4k$ lies entirely below the x-axis.

(i) Let $y = 3x^2 + x + k$, where $y > 0$

Discriminant = $1^2 - 4(3)(k)$

$= 1 - 12k$

$1 - 12k < 0$

$12k > 1$

$k > \frac{1}{12}$

(ii) Discriminant = $1^2 - 4(k)(4k)$

$= 1 - 16k^2$

$1 - 16k^2 < 0$ *and* $k < 0$

$16k^2 > 1$

$k^2 > \frac{1}{16} \Leftrightarrow k > \frac{1}{4}$ *or* $k < -\frac{1}{4}$

However, k must be negative as it is the coefficient of x^2 for the graph to be ∩-shaped.

$k < -\frac{1}{4}$ (rej. $k > \frac{1}{4}$, $k < 0$ into account)

Real-Life Applications of Quadratic Functions

Quadratic functions can be used as models in real-life applications.

Examples: A firework is launched from a platform 10m above the ground.

Its height h after t seconds is given by: $h = -4t^2 + 40t + k$, where k is a constant.

(a) Find the range of values of k such that the firework never reaches a height of 120m.

(b) If $k = 10$, express h in the form of $h = a(t - p)^2 + q$.

Hence find the maximum height and the time at which it occurs.

(c) Explain how your answer in **part(b)** relates to your answer in **part(a)**.

(a) Let $h = 120$

$-4t^2 + 40t + k = 120$

$-4t^2 + 40 + (k - 120) = 0$

Discriminant $= 40^2 - 4(-4)(k - 120)$

$= 1600 + 16k - 1920$

$= 16k - 320$

For the firework to never reach 120m, this equation must have no real solutions.

$16k - 320 < 0$

$16k < 320$

$k < 20$

(b) $h = -4t^2 + 40t + 10$

$= -4(t^2 - 10t) + 10$

$= -4[(t - 5)^2 - (-5)^2] + 10$

$= -4(t - 5)^2 + 4(-5)^2 + 10$

$= -4(t - 5)^2 + 110$

turning point(5, 110), therefore 110m is the max height at time 5 seconds when $k = 10$.

(c) From (a), $k < 20$

From (b), When $k = 10, max\ h = 110$

If $k = 20,\ max\ h = 120$

The discriminant condition of $k < 20$ ensures that the maximum height is always less than 120m.

This can be seen from part(b) where when $k = 10 < 20$, the maximum height is 110m.

A petrol station finds that when the price of petrol is x dollars per litre, the number of litres sold per day is $600 - 20x$.

(a) If k is a fixed daily operating cost, and P is the daily profit, find an equation in P.
(b) Show that the maximum profit is $4500 - k$.
(c) **Hence,** find the range of values of k where the petrol station makes no profit.
(d) Verify your answer by using **part(a)**.

(a) sales from petrol $= x(600 - 20x)$
$= 600x - 20x^2$

$P = 600x - 20x^2 - k$
$= -20x^2 + 600x - k$

(b) $P = -20x^2 + 600x - k$
$= -20(x^2 - 30x) - k$
$= -20[(x - 15)^2 - 15^2] - k$
$= -20(x - 15)^2 + 20(15^2) - k$
$= -20(x - 15)^2 + (4500 - k)$

The maximum P is $(4500 - k)$ as turning point is $(15,\ 4500 - k)$

(c) For the petrol station to make no profit, P must be non-positive.
$4500 - k \leq 0$
$k \geq 4500$

(d) From part(a), $P = -20x^2 + 600x - k$

Discriminant $= 600^2 - 4(-20)(-k)$
$= 360000 - 80k$

For $P \leq 0$, the graph must touch the x-axis or lie completely below the x-axis.

$D \leq 0$
$360000 - 80k \leq 0$
$80k \geq 360000$
$k \geq 4500\ (shown)$

Two drones are flying vertically above the ground. Their height in metres after t seconds is modelled:

$$h_1 = 2t^2 - 16t + 35$$

$$h_2 = 4t^2 - 24t + 41$$

(a) Show that the drones are at the same height in the air twice.

(b) Find the times when the drones are at the same height.

(c) Find an equation in s: the vertical separation between the two drones.

(d) Determine whether the drones will collide. Give a reason.

(a) Sub h_1 into h_2:

$$2t^2 - 16t + 35 = 4t^2 - 24t + 41$$

$$2t^2 - 8t + 6 = 0$$

$$t^2 - 4t + 3 = 0$$

$$\begin{aligned}\text{Discriminant} &= (-4)^2 - 4(1)(3) \\ &= 16 - 12 \\ &= 4 \\ &> 0\end{aligned}$$

When equating the two equations together, $D = 4 > 0$, which means that there are two real roots, hence the drones are at the same height in the air at two time intervals.

(b) $$t^2 - 4t + 3 = 0$$

$$(t - 3)(t - 1) = 0$$

$$t - 3 = 0 \text{ or } t - 1 = 0$$

$$t = 3 \text{ or } t = 1$$

The drones are at the same height at 1 second and 3 seconds.

(c) vertical separation, $s = |h_1 - h_2|$ →Absolute value as s must be non-negative

$$= |(2t^2 - 16t + 35) - (4t^2 - 24t + 41)|$$

$$= |-2t^2 + 8t - 6|$$ →Factor out -1

$$= |2t^2 - 8t + 6|$$

(d) At $t = 3$ and $t = 1$, the drones are at the same vertical height. However, we are unsure whether they will collide as we are not given the horizontal positions of the drones, hence we do not know the horizontal distance between the two. They can only collide if the horizontal distance between the two drones $= 0$.

For visual learners, here are the graphs in graphing software for each real-life application question:
Fireworks: https://www.desmos.com/calculator/exsr2hznmh
Profit Petrol: https://www.desmos.com/calculator/xgiwyjwnli
Drones: https://www.desmos.com/calculator/4cbtkv0y90

End of Notes