

## Simple Harmonic Motion

- We know that a net force acting on an object causes the object to accelerate (Newton's laws of motion). In many cases that motion is in one direction, but there are cases where the direction of the net force alternates back and forth repeatedly which causes the object to oscillate.
- **Periodic motion** is any motion that repeats in equal intervals of time. This is a broad category which could include things like a person on a swing, a person in a rocking chair, a heartbeat or pulse, waves crashing on the shore, the tides alternating between high and low, and any example of uniform circular motion like wheels rotating or planets orbiting the sun.
- **Simple harmonic motion** specifically refers to the periodic motion of an object that occurs due to a **restoring force** which is proportional to the distance of the object from its equilibrium position. The most common examples are a mass attached to a spring and a simple pendulum.
- The **period** is the duration of one oscillation or cycle, how long it takes the object to return to its original position.
- The **frequency** is the number of oscillations or cycles per second, which is the inverse of the period.

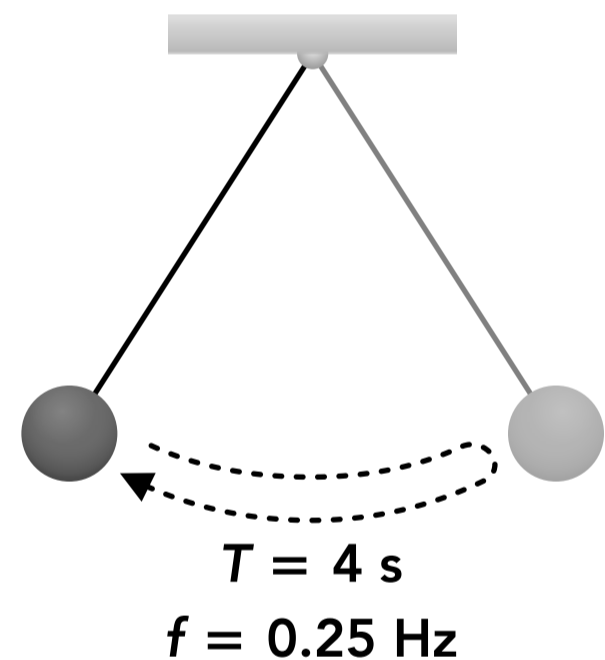
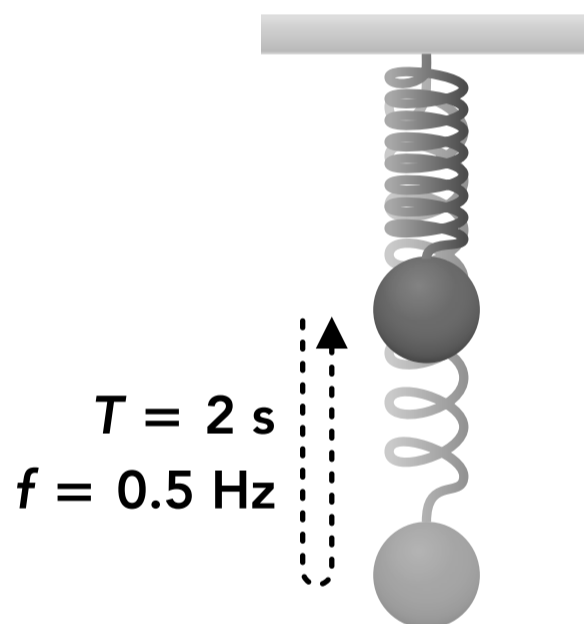
Variables		SI Unit
$T$	period	s
$f$	frequency	Hz = $\frac{\text{cycles}}{\text{s}}$
$A$	amplitude	m
$x, y$	position	m
$v$	velocity	$\frac{\text{m}}{\text{s}}$
$a$	acceleration	$\frac{\text{m}}{\text{s}^2}$

Objects in simple harmonic motion (and other periodic motion) repeat their motion with the same period and frequency

Frequency

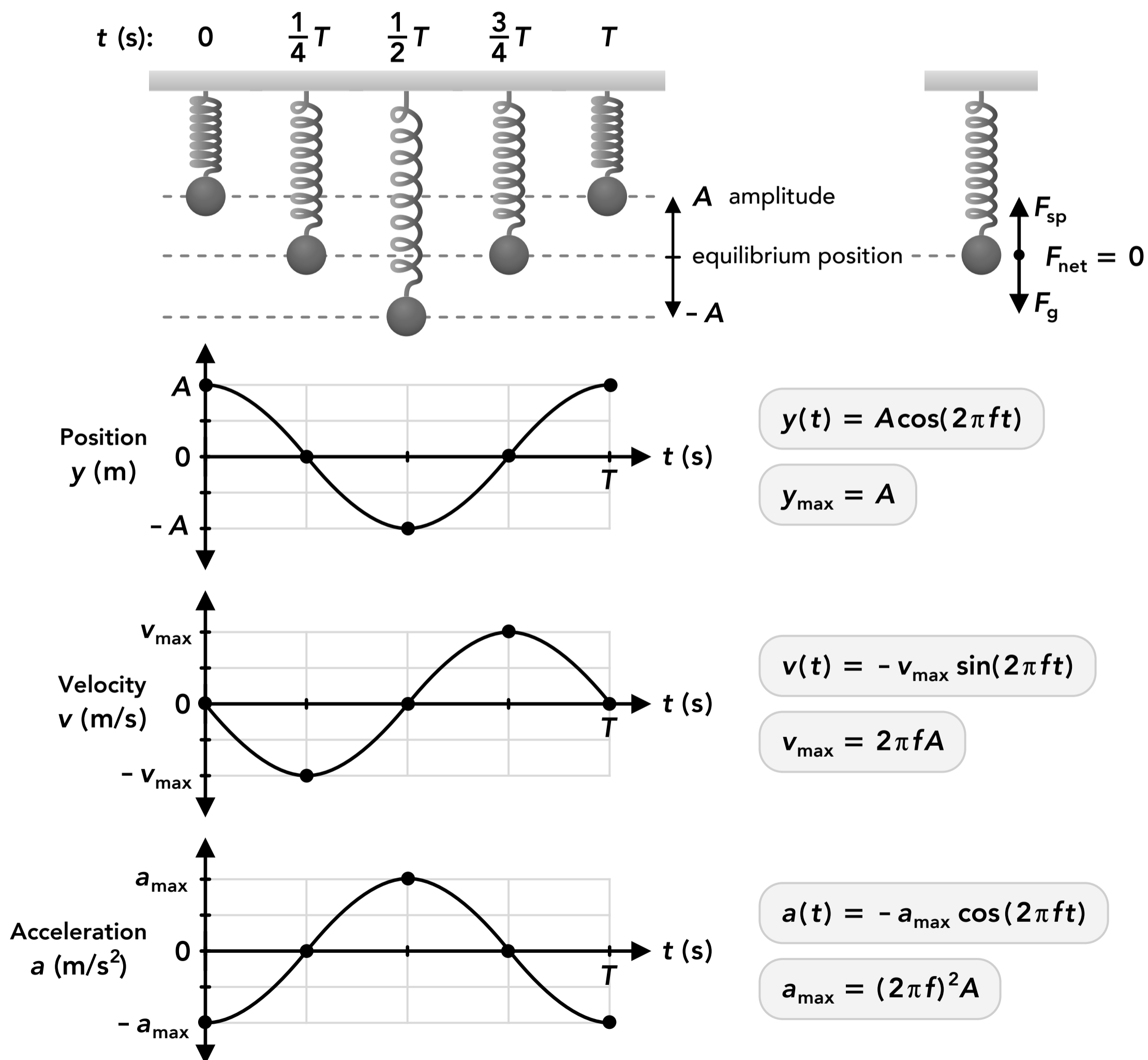
$$f = \frac{1}{T}$$

$f$ : frequency (Hz, cycles/s)  
 $T$ : period (s)



- The **equilibrium position** of a simple harmonic motion is the position where **the net force on the object is zero** and the acceleration is zero. If the object is placed in the equilibrium position and released it will not move. For a horizontal mass-spring system this is the position where the spring is at its original length and the spring force is zero. For a vertical mass-spring system this is the stretched length where the upwards spring force is equal to the downwards gravitational force. For a pendulum this is lowest position where the pendulum is vertical.
- The **amplitude** of a simple harmonic motion is the distance between the equilibrium position and one end of the oscillation. The distance between both ends of the oscillation is twice the amplitude.
- If we graph the position of the object over time the graph is **sinusoidal** (it's a sine wave, with some phase shift).
- For any motion, the value of the velocity graph is the slope of the position graph at any point in time, and the value of the acceleration graph is the slope of the velocity graph at any point in time. This results in all three graphs being sinusoidal as seen below.
- We can describe the position, velocity and acceleration over time using the wave equations below which depend on the amplitude, frequency and time.

Graphs of the position, velocity and acceleration of a mass-spring system for one oscillation (one period)



## Mass-Spring Systems

- One example of simple harmonic motion is a mass attached to a spring oscillating horizontally or vertically.
- For a horizontal mass-spring system the spring force acts as the restoring force.
- For a vertical mass-spring system a combination of the spring force and the gravitational force act as the restoring force.
- We're going to assume there are no friction or drag forces so the system does not lose any energy and continues oscillating forever. This is called the "undamped" case.
- The period of a mass-spring oscillation depends on the mass of the object and the spring constant (we assume the spring is massless). Notice that it does not depend on the amplitude.

Variables		SI Unit
$T$	period	s
$f$	frequency	Hz = $\frac{\text{cycles}}{\text{s}}$
$A$	amplitude	m
$m$	mass	kg
$k$	spring constant	$\frac{\text{N}}{\text{s}}$
$U_{\text{sp}}$	spring potential energy	J
$K$	kinetic energy	J

Period of a mass-spring oscillation

$$T_{\text{sp}} = 2\pi \sqrt{\frac{m}{k}}$$

Frequency of a mass-spring oscillation

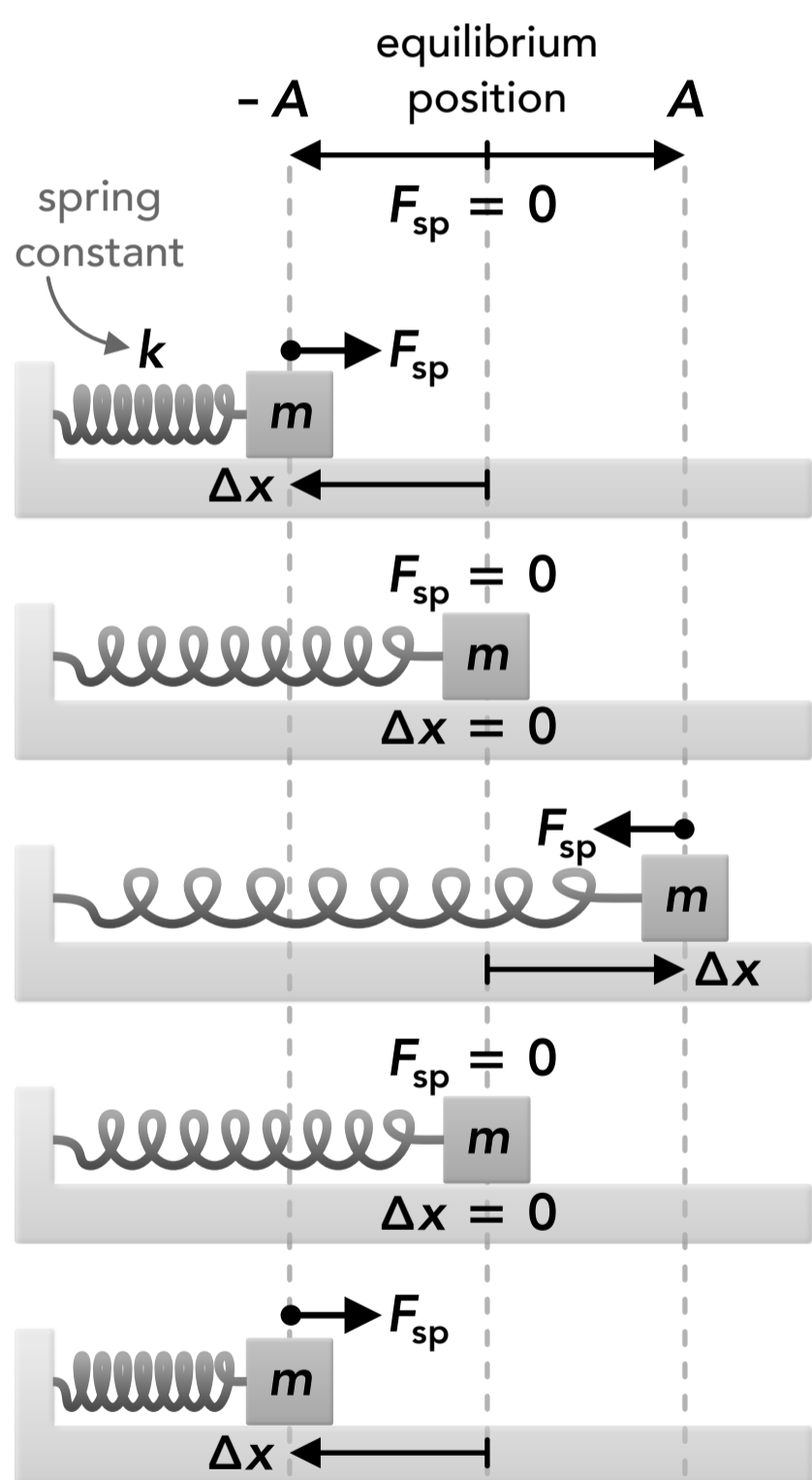
$$f_{\text{sp}} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

Maximum velocity of a mass-spring oscillation

$$v_{\text{max}} = A \sqrt{\frac{k}{m}}$$

- For a **horizontal** mass-spring system the equilibrium position is at the unstretched spring position where the displacement is zero and the spring force is zero.
- The restoring force is always the spring force acting on the mass which alternates directions when the spring switches between being stretched or compressed. The spring force is at its maximum magnitude when the object is at the maximum displacement from the equilibrium position.
- The velocity of the mass is at its maximum magnitude at the equilibrium position and zero at the maximum displacement when the object is momentarily at rest while it reverses direction.
- The acceleration of the mass depends on the spring force and is at its maximum magnitude at the maximum displacement and zero at the equilibrium position.
- The spring potential energy depends on the displacement and is at its maximum value at the maximum displacement and zero at the equilibrium position.
- The kinetic energy depends on the velocity of the mass and is at its maximum value at the equilibrium position and zero at the maximum displacement.

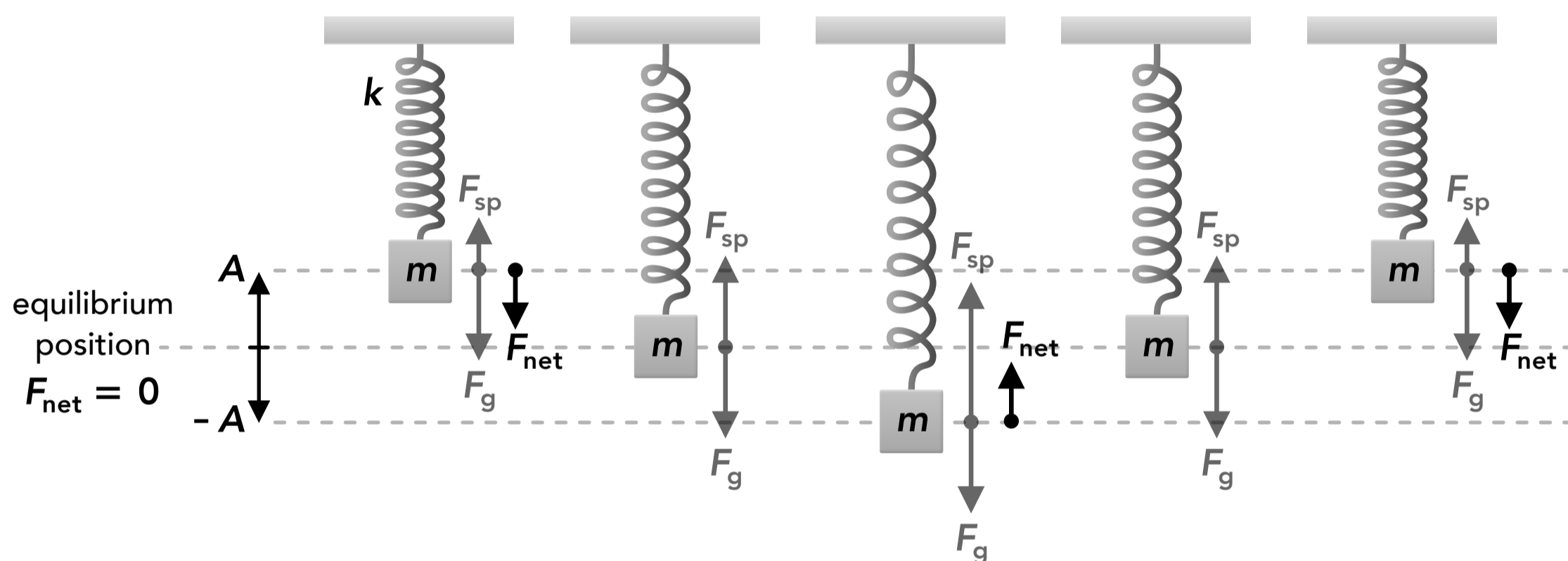
Horizontal mass-spring system (assuming no friction)



	Spring force	Spring potential energy	Kinetic energy		
	$F_{sp} = k\Delta x$	$U_{sp} = \frac{1}{2}k\Delta x^2$	$K = \frac{1}{2}mv^2$		
$x$	$v$	$a$	$F_{sp}$	$U_{sp}$	$K$
$x = -A$	$v = 0$	$a_{\max}$	$F_{sp \max}$	$U_{sp \max}$	$K = 0$
$x = \text{eq}$	$v_{\max}$	$a = 0$	$F_{sp} = 0$	$U_{sp} = 0$	$K_{\max}$
$x = A$	$v = 0$	$a_{\max}$	$F_{sp \max}$	$U_{sp \max}$	$K = 0$
$x = \text{eq}$	$v_{\max}$	$a = 0$	$F_{sp} = 0$	$U_{sp} = 0$	$K_{\max}$
$x = -A$	$v = 0$	$a_{\max}$	$F_{sp \max}$	$U_{sp \max}$	$K = 0$

- The addition of a gravitational force and gravitational potential energy makes a vertical mass-spring system slightly more complex than a horizontal mass-spring system, but the motion behaves the same way.
- For a **vertical** mass-spring system the equilibrium position is where the upwards spring force and the downwards gravitational force are equal in magnitude so the net force on the mass is zero. The spring is already stretched some initial displacement due to the gravitational force on the mass. Therefore the actual spring force is not based on the object's displacement from the equilibrium position but from the displacement from the original unstretched length of the spring. We're going to assume the spring is always stretched some amount.
- The restoring force is a combination of the spring force (always upwards) and the gravitational force (always downwards). When the spring is at the equilibrium position the two forces are equal and the net force is zero. When the mass is above the equilibrium position the spring force is decreased and the net force is downwards. When the mass is below the equilibrium position the spring force is increased and the net force is upwards.
- The position, velocity, acceleration, net force and kinetic energy are at their maximum and zero values at the same points in the motion as a horizontal mass-spring system.
- The spring potential energy is at its maximum at the lowest position and its minimum at the highest position.
- The gravitational potential energy is at its maximum at the highest position and its minimum at the lowest position.

Vertical mass-spring system

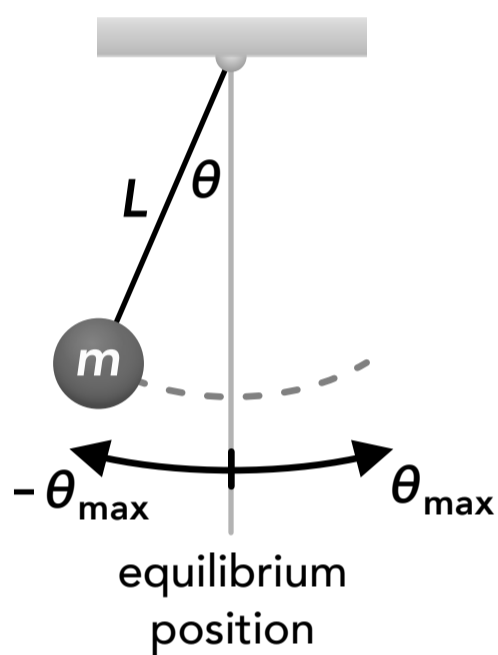


$y$	$y = A$	$y = \text{eq}$	$y = -A$	$y = \text{eq}$	$y = A$
$v$	$v = 0$	$v_{\text{max}}$	$v = 0$	$v_{\text{max}}$	$v = 0$
$a$	$a_{\text{max}}$	$a = 0$	$a_{\text{max}}$	$a = 0$	$a_{\text{max}}$
$F_{\text{sp}}$	$F_{\text{sp min}}$	$F_{\text{sp}}$	$F_{\text{sp max}}$	$F_{\text{sp}}$	$F_{\text{sp min}}$
$F_g$	$F_g$	$F_g$	$F_g$	$F_g$	$F_g$
$U_{\text{sp}}$	$U_{\text{sp min}}$	$U_{\text{sp}}$	$U_{\text{sp max}}$	$U_{\text{sp}}$	$U_{\text{sp min}}$
$U_g$	$U_g \text{ max}$	$U_g$	$U_g \text{ min}$	$U_g$	$U_g \text{ max}$
$K$	$K = 0$	$K_{\text{max}}$	$K = 0$	$K_{\text{max}}$	$K = 0$

## Pendulums

- Another example of simple harmonic motion is a simple pendulum which is a mass hanging from a string (or other long object) which swings back and forth.
- The restoring force for a pendulum is the component of the gravitational force which acts tangentially to the circular path of the mass.
- We're going to look at a **simple pendulum** which includes a few assumptions: the rope is massless and **the maximum angle of the pendulum (the amplitude) is small** ( $< \sim 10^\circ$ ). This means that the restoring force will be approximately proportional to the displacement from the equilibrium position.
- The period of a pendulum depends on the length of the pendulum and the acceleration due to gravity. Note that it does not depend on the mass or the amplitude.

Variables		SI Unit
$T$	period	s
$f$	frequency	Hz = $\frac{\text{cycles}}{\text{s}}$
$\theta$	angle	rad
$L$	length	m
$U_g$	grav. potential energy	J
$K$	kinetic energy	J



Period of a pendulum oscillation

$$T_p = 2\pi \sqrt{\frac{L}{g}}$$

Frequency of a pendulum oscillation

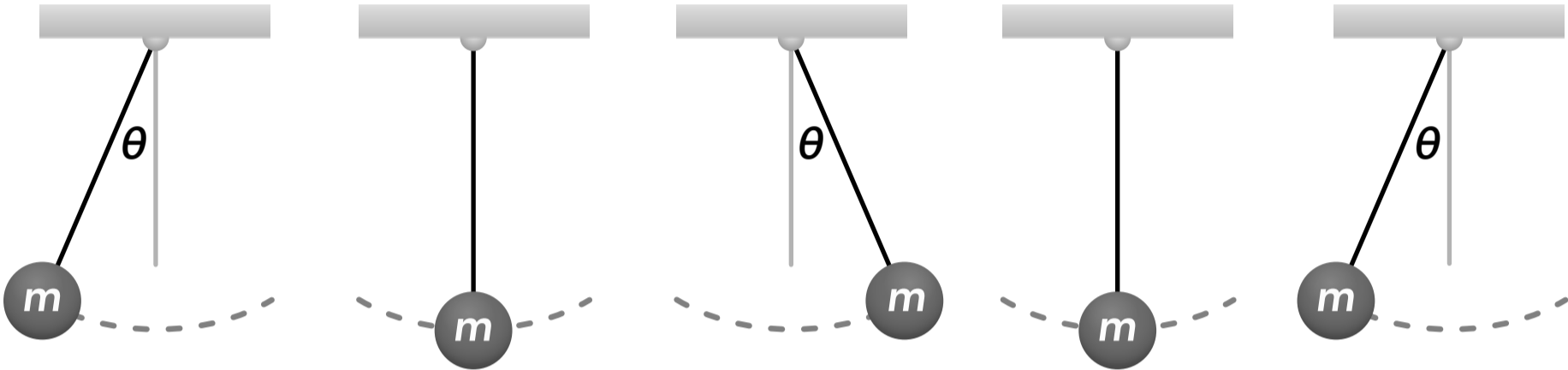
$$f_p = \frac{1}{2\pi} \sqrt{\frac{g}{L}}$$

Maximum velocity of a pendulum oscillation

$$v_{\max} = \theta_{\max} \sqrt{gL}$$

- The equilibrium position for a pendulum is when the mass is at the lowest height.
- The velocity of the mass is at its maximum magnitude at the equilibrium position and zero at the maximum displacement when the mass is momentarily at rest while it reverses direction.
- The tangential and angular acceleration of the mass is at its maximum at the maximum displacement and zero at the equilibrium position.
- The gravitational potential energy is at its maximum when the mass is at the maximum displacement (which is the point of maximum height) and is at its minimum at the equilibrium position (lowest height).
- The kinetic energy depends on the velocity of the mass and is at its maximum at the equilibrium position and zero at the maximum displacement.

Simple pendulum (massless rope and small maximum angle)



$\theta$	$-\theta_{\max}$	$\theta = 0$	$\theta_{\max}$	$\theta = 0$	$-\theta_{\max}$
$y$	$y_{\max}$	$y_{\min}$	$y_{\max}$	$y_{\min}$	$y_{\max}$
$v$	$v = 0$	$v_{\max}$	$v = 0$	$v_{\max}$	$v = 0$
$a_t$	$a_{t \max}$	$a_t = 0$	$a_{t \max}$	$a_t = 0$	$a_{t \max}$
$\alpha$	$\alpha_{\max}$	$\alpha = 0$	$\alpha_{\max}$	$\alpha = 0$	$\alpha_{\max}$
$U_g$	$U_{g \max}$	$U_{g \min}$	$U_{g \max}$	$U_{g \min}$	$U_{g \max}$
$K$	$K = 0$	$K_{\max}$	$K = 0$	$K_{\max}$	$K = 0$