2: Equations and Inequalities

Simultaneous Equations

We have learnt how to solve simultaneous equations in *Lower Secondary G3 Math Revision Notes.* For a quick recap, there's two ways to solve it: Elimination and Substitution. This applied to solving simultaneous linear equations. However, in aMath, we will solve **one linear** equation and **one non-linear** equation simultaneously, which means the substitution method works best.
*The algebraic solutions of two equations are the points of intersection between the two graphs.

Examples: Solve the following simultaneous equations:

$$x^0 = xy - 5 \ and \ y - x = -1$$

$x^0 = xy - 5$ −− (1)
$y - x = -1$ −− (2)

From (1), $1 = xy - 5$
From (2), $y = x - 1$

Substitute (2) into (1): $1 = x(x - 1) - 5$
$= x^2 - x - 5$

$x^2 - x - 6 = 0$
$(x - 3)(x + 2) = 0$
$x = 3 \ or \ x = -2$

When $x = 3$: $y = 3 - 1$
$= 2$

When $x = -2$: $y = -2 - 1$
$= -3$

Solutions are $x = 3, \ y = 2$
or $x = -2, \ y = -3$

* Solutions of one linear equation and one non-linear equation come in two pairs of x and its corresponding y value. Answers should not be given as one pair of 2 x values and one pair of 2 y values, instead should be given in (x_1, y_1) and (x_2, y_2) respectfully.

Find the points of intersection of graph $y = x^2 - 3x + 1$ and $y = 2 - x$.

$y = x^2 - 3x + 1$ -- (1)
$y = 2 - x$ ------- (2)

Substitute (1) into (2): $x^2 - 3x + 1 = 2 - x$
$x^2 - 2x - 1 = 0$

Using quadratic formula,

$$x = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-1)}}{2(1)}$$
$$= \frac{2\pm\sqrt{8}}{2}$$
$$= 2.41 \ or \ -0.41 \ (to\ 3s.f.)$$

When $x = 2.41$, $y = 2 - 2.41$
$= -0.41$

When $x = -0.41$, $y = 2 - (-0.41)$
$= 2.41$

The points of intersection of $y = x^2 - 3x + 1$ and $y = 2 - x$
is $(2.41, -0.41)$ and $(-0.41, 2.41)$.

A rectangular field has a perimeter of $50m$ and an area of $144m^2$.
By letting the unknowns be algebraic constants, find the dimensions of the field.

Let x be the length of the field while y is the breadth of the field.
$2(x + y) = 50$
$x + y = 25$
$x = 25 - y$ -- (1)

$xy = 144$ -- (2)

Substitute (1) into (2): $(25 - y)(y) = 144$
$25y - y^2 = 144$
$-y^2 + 25y - 144 = 0$
$(16 - y)(y - 9) = 0$
$y = 16 \ or \ y = 9$

When $y = 16$, $x = 9$
When $y = 9$, $x = 16$

Therefore, the dimensions of the rectangular field is $16m \times 9m$.

Nature of Roots of Quadratic Equations

In *Chapter 1, we found out how to use the discriminant to find how many real roots / x-intercepts an equation had.* Similarly, we can use the discriminant to find how many points of intersection / solutions there are for a pair of two equations by substituting them into one another.

*If a straight line intersects a curve at only one point (discriminant $= 0$), the line is considered tangent to the curve.

For example,

Show that the graphs $y = \frac{1}{2}x^2 - \frac{3}{2}x + 1$ and $y = \frac{3}{2}x - \frac{1}{2}$ intersects at two distinct points.

Substitute $y = \frac{1}{2}x^2 - \frac{3}{2}x + 1$ into $y = \frac{3}{2}x - \frac{1}{2}$:

$$\frac{1}{2}x^2 - \frac{3}{2}x + 1 = \frac{3}{2}x - \frac{1}{2}$$
$$\frac{1}{2}x^2 - 3x + \frac{3}{2} = 0$$

Discriminant $= (-3)^2 - 4(\frac{1}{2})(\frac{3}{2})$
$= 6 > 0$

Since discriminant > 0, the graphs $y = \frac{1}{2}x^2 - \frac{3}{2}x + 1$ and $y = \frac{3}{2}x - \frac{1}{2}$ intersects at two distinct points.

Explain the *relationship* between graphs $y = x^2 - 4x + \frac{7}{4}$ and $y = -x - \frac{1}{2}$.

Substitute $y = x^2 - 4x + \frac{7}{4}$ into $y = -x - \frac{1}{2}$:

$$x^2 - 4x + \frac{7}{4} = -x - \frac{1}{2}$$
$$x^2 - 3x + \frac{9}{4} = 0$$

Discriminant $= (-3)^2 - 4(1)(\frac{9}{4})$
$= 0$

Since discriminant $= 0$, the graphs $y = x^2 - 4x + \frac{7}{4}$ and $y = -x - \frac{1}{2}$ intersect at only one point. As such, the line $y = -x - \frac{1}{2}$ is tangent to the curve of $y = x^2 - 4x + \frac{7}{4}$.

The same discriminant rules apply when solving questions as such. If $D < 0$, the two lines do not intersect. If $D = 0$, the two lines intersect at one point. If $D \geq 0$, the two lines may intersect at one or two points. If $D > 0$, the two lines intersect at two distinct points. The only difference here is that we substitute the two lines into each other. (Basically, the x-axis, $y = 0$, will be replaced with the other linear equation given.)

Two graphs of $y = x^2 - 2x + k$ and $y = x + 1$ lie on a Cartesian plane.
(a) Find the range of values of k when there's at least one point of intersection.
(b) **From (a),** find the values of k such that there is only one point of intersection.
(c) A new line of $y = kx^2 - 4kx + 3$ replaces $y = x^2 - 2x + k$.
Find the values of k such that the two graphs do not intersect.

(a) Substitute $y = x^2 - 2x + k$ into $y = x + 1$

$$x^2 - 2x + k = x + 1$$
$$x^2 - 3x + (k - 1) = 0$$

For there to be at least one point of intersection,

$$\text{discriminant} \geq 0$$
$$(-3)^2 - 4(1)(k - 1) \geq 0$$
$$9 - 4k + 4 \geq 0$$
$$13 - 4k \geq 0$$
$$4k \leq 13$$
$$k \leq \frac{13}{4}$$

$k \leq \frac{13}{4}$ when there is at least one point of intersection between graphs $y = x^2 - 2x + k$ and $y = x + 1$.

(b) $k = \frac{13}{4}$ when there is one point of intersection between graphs $y = x^2 - 2x + k$ and $y = x + 1$.

(c) Substitute $y = kx^2 - 4kx + 3$ into $y = x + 1$

$$kx^2 - 4kx + 3 = x + 1$$
$$kx^2 + (-4k - 1)x + 2 = 0$$

For the graphs not to intersect,

$$\text{discriminant} < 0$$
$$(-4k - 1)^2 - 4(k)(2) < 0$$
$$16k^2 + 8k + 1 - 8k < 0$$
$$16k^2 + 1 < 0$$
$$16k^2 < -1$$
$$k^2 < \frac{-1}{16}$$

There are no real values of k such that the two graphs do not intersect since $k^2 \geq 0$ for all real values of k.

Show that the roots of the equation $x^2 + (p+1)x = 7 - 2p$ are real for all real values of p.

$x^2 + (p+1)x + (2p-7) = 0$

$$\begin{aligned}\text{Discriminant} &= (p+1)^2 - 4(1)(2p-7)\\ &= p^2 + 2p + 1 - 10p + 28\\ &= p^2 - 8p + 29\\ &= (p-4)^2 + 13\end{aligned}$$

Since $(p-4)^2 \geq 0$

$(p-4)^2 + 13 \geq 13$

$D \geq 13 > 0$

Since discriminant > 0, the roots of the equation are real for all values of p.

Given $p > \frac{21}{8}$, show that the equation $2x^2 + p = 2(x-1)$ has no real roots.

$2x^2 - 2x + (p-2) = 0$

$$\begin{aligned}\text{Discriminant} &= (-2)^2 - 4(2)(p-2)\\ &= 4 - 8p + 16\\ &= 20 - 8p\end{aligned}$$

Since $p > \frac{21}{8}$,

$-8p < -21$

$20 - 8p < -1$

$D < -1 < 0$

Since discriminant < 0, the equation has no real roots.

Explain whether the line $y = -6x - 3$ intersects the curve $y = mx^2 + 2$, where $m < 1$.

Substitute $y = -6x - 3$ into $y = mx^2 + 2$:

$-6x - 3 = mx^2 + 2$

$mx^2 + 6x + 5 = 0$

$$\begin{aligned}\text{Discriminant} &= (6)^2 - 4(m)(5)\\ &= 36 - 20m\end{aligned}$$

Since $m < 1$,

$-20m > -20$

$36 - 20m > 16$

$D > 16 \geq 0$

Since discriminant ≥ 0, the line will intersect the curve when $m < 1$.

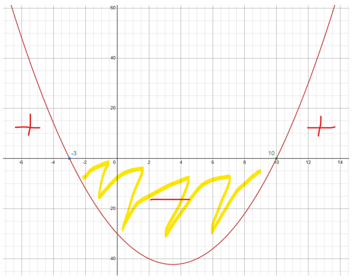
Solving Quadratic Inequalities

Solving quadratic inequalities follows a 4-step process:

- Shift everything to the LHS so RHS = 0
- Consider the inequality as a quadratic function:
  – If possible to factorize, factorize to find the x-intercepts.
  – If not, use quadratic formula to find the x-intercepts.
- Sketch the quadratic graph
- Deduce the solution set

Examples: RHS already = 0.

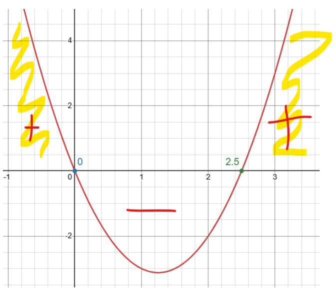
$(x + 3)(x - 10) \leq 0$



As such, $-3 \leq x \leq 10$ for $(x + 3)(x - 10) \leq 0$ to be satisfied.

$2x^2 - 5x > 0$

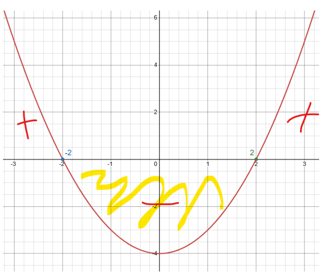
$x(2x - 5) > 0$



As such, $x < 0$ or $x > \frac{5}{2}$ for $2x^2 - 5x > 0$ to be satisfied.

$x^2 - 4 < 0$

$(x - 2)(x + 2) < 0$



As such, $-2 < x < 2$ for $x^2 - 4 < 0$ to be satisfied.

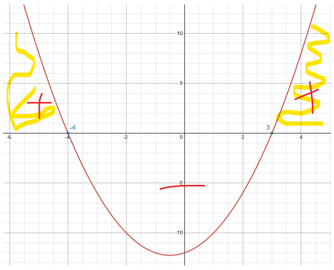
Examples: RHS not already = 0.

$$(2 - x)^2 \geq 16 - 5x$$

$$4 - 4x + x^2 \geq 16 - 5x$$

$$x^2 + x - 12 \geq 0$$

$$(x + 4)(x - 3) \geq 0$$



As such, $x \leq -4$ or $x \geq 3$ for $(2 - x)^2 \geq 16 - 5x$ to be satisfied.

$$x^2 + 3x \leq 21$$

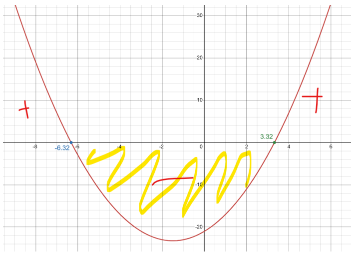
$$x^2 + 3x - 21 \leq 0$$

Considering $x^2 + 3x - 21 = 0$:

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-21)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{93}}{2}$$

$$= -6.32,\ 3.32\ (to\ 3s.f.)$$



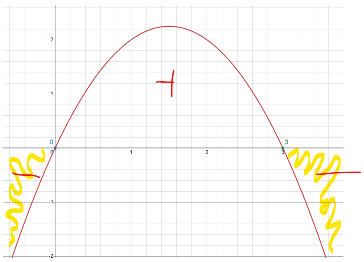
As such, $-6.32 \leq x \leq 3.32$ for $x^2 + 3x \leq 21$ to be satisfied.

$$x(5 - x) < 2x$$

$$5x - x^2 < 2x$$

$$-x^2 + 3x < 0$$

$$-x(x - 3) < 0$$



As such, $x < 0$ or $x > 3$ for $x(5 - x) < 2x$ to be satisfied.

Examples: Range of values

Find the range of values of k such that $x^2 + (k + 1)x + (k^2 - 4) = 0$ has 2 real, distinct roots.

For equation to have two distinct and real roots,

$$\text{discriminant} > 0$$

$$(k + 1)^2 - 4(1)(k^2 - 4) > 0$$

$$k^2 + 2k + 1 - 4k^2 + 16 > 0$$

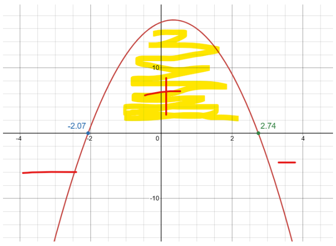
$$-3k^2 + 2k + 17 > 0$$

Considering $-3k^2 + 2k + 17 = 0$:

$$k = \frac{-2 \pm \sqrt{2^2 - 4(-3)(17)}}{2(-3)}$$

$$= \frac{-2 \pm \sqrt{208}}{-6}$$

$$= -2.07,\ 2.74\ (to\ 3\ s.f.)$$

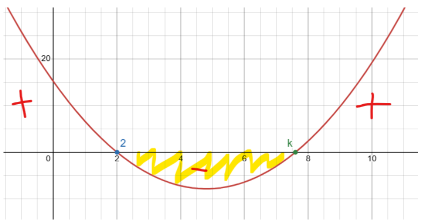


As such, $-2.07 < k < 2.74$ such that $x^2 + (k + 1)x + (k^2 - 4) = 0$ has 2 real, distinct roots.

Examples: Working backwards

The equation of a quadratic curve is $y = 2x^2 + px + 16$. Given $y < 0$ only when $2 < x < k$, find the value of p and k.

Since $2 < x < k$,



$$a(x - 2)(x - k) < 0$$

From the question, $2x^2 + px + 16 < 0$ when $a(x - 2)(x - k) < 0$. Therefore,

$2x^2 + px + 16 = 2(x - 2)(x - k)$ $[a = 2]$

Comparing constants: $16 = 4k$

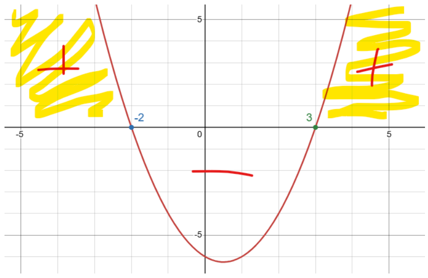
$$k = 4$$

Comparing coefficients of x: $p = -2k - 4$

$$p = -2(4) - 4 = -12$$

Find the value of a and b for which $3x^2 + a > bx$ is only satisfied by $x < -2$ or $x > 3$.

Since $x < -2$ or $x > 3$,



$k(x + 2)(x - 3) > 0$, where $k = 3$

$$3(x + 2)(x - 3) > 0$$

$$3(x^2 - x - 6) > 0$$

$$3x^2 - 3x - 18 > 0$$

$$3x^2 - 18 > 3x$$

$$a = -18,\ b = 3$$