Linearized Gravity: An Intuitive Introduction

This lesson covers **linearized gravity**, the framework for describing *weak* gravity in general relativity. Linearized gravity allows us to describe two main applications: the **Newtonian limit** and **gravitational waves**. We'll also be able to describe frame dragging, gravitational wave detectors and other applications within this framework.

I'd recommend working through this *after Lesson 12* of the course. While linearized gravity is a stand-alone topic, we will use many of the mathematical tools from previous lessons, so having an understanding of those first should be beneficial.

Lesson Contents:

1. The General Framework of Linearized Gravity

General relativity is a non-linear theory, and solving Einstein's equations (analytically) is only possible in very special situations. Linearized gravity falls under these "special situations" - it is a general **framework** that can be used to solve the field equations for many different spacetimes under the assumption that spacetime curvature is *weak*.

In this context, "weak" means that all *non-linear* terms in the Einstein field equations are so small they can be taken as zero, which makes the field equations themselves **linear**. In the language of differential equations, this usually translates to "much simpler to solve".

For this section, we'll begin by discussing some general aspects and mathematical rules of linearized gravity. If you don't care about the details of the calculations behind, for example, the Einstein field equations, all the important results are highlighted, so it is possible to just glance over the following sections. After developing the basic mathematical framework, we'll discuss gauge transformations and some applications.

1.1. The Rules of Linearized Gravity

The starting point for linearized gravity is to mathematically describe a *weakly curved* spacetime. The way to do this is to pick a **background spacetime** and add a correction to it, which represents how curvature makes the "full" metric deviate from that background.

In the case of linearized gravity, we usually pick the background as the *flat* **Minkowski spacetime** of special relativity described by the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. The metric of the "full" curved spacetime is then this background metric *plus* a small correction called a **metric perturbation**, which we denote by $h_{\mu\nu}$. This perturbation describes how our full metric deviates from the background, in this case flat spacetime:

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}}$$

So, the key idea behind this framework is to split the spacetime into a *background* and an additional *field* on top of it. This field, the metric perturbation, is what contains gravity.

An important assumption here is that the background metric is expressed in Cartesian coordinates, so it really has the form $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, and importantly, $\partial_\alpha \eta_{\mu\nu} = 0$. If needed, we can always perform a coordinate transformation to some other coordinate system at the end, but the intermediate calculations we'll do assume that the background metric is constant. The linearized gravity metric can then be written in the general form:

$$g_{\mu\nu} = \begin{pmatrix} -1 + h_{00} & h_{01} & h_{02} & h_{03} \\ h_{01} & 1 + h_{11} & h_{12} & h_{13} \\ h_{02} & h_{12} & 1 + h_{22} & h_{23} \\ h_{03} & h_{13} & h_{23} & 1 + h_{33} \end{pmatrix} \sim \begin{pmatrix} -1 + h_{00} & h_{0i} \\ h_{0i} & \delta_{ij} + h_{ij} \end{pmatrix}$$

So far, this is just a way to rewrite the metric. Linearized gravity is the study of *weak* gravity, so it additionally assumes that the **perturbation $h_{\mu\nu}$ is small**. Mathematically, 'small' means that anything *non-linear* in the perturbation is taken as zero - so, schematically $h^2 \sim 0$, $h^3 \sim 0$ and so on. Because derivatives of the perturbation also contribute to gravity, these are also taken to be small, so $(\partial h)^2 \sim 0$, $(\partial h)^3 \sim 0$ as well as anything of the form $h \partial h \sim 0$.

The Assumptions of Linearized Gravity

- **The metric tensor** can be written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.
- **The metric perturbations are small**, which means any terms containing higher powers in $h_{\mu\nu}$ are zero - so, anything like $h_{\mu\nu} h_{\alpha\beta} \approx 0$.
- **The metric perturbation varies slowly**, so derivatives of $h_{\mu\nu}$ are also small, and any terms of the form $h_{\mu\nu} \partial_\sigma h_{\alpha\beta} \approx 0$ or $\partial_\lambda h_{\mu\nu} \partial_\sigma h_{\alpha\beta} \approx 0$.

These rules essentially remove all non-linearities from general relativity. An important consequence of this is that indices are raised and lowered with the background Minkowski metric (if we were to raise them with the full metric, we'd get non-linear terms). The perturbation $h_{\mu\nu}$ behaves more like an "external" field on top of a flat background spacetime, and any tensor operations are done strictly with the background metric.

Inverse Metric & Christoffel Symbols In The Linearized Limit

The first couple of things these rules allow us to work out are the inverse metric and the Christoffel symbols. We can find the inverse metric from the definition that $g_{\mu\nu}g^{\mu\nu} = 4$, which can also be written in terms of the Minkowski metric as:

$$g_{\mu\nu}g^{\mu\nu} = \eta_{\mu\nu}\eta^{\mu\nu}$$

Inserting the metric in the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we find:

$$(\eta_{\mu\nu} + h_{\mu\nu})g^{\mu\nu} = \eta_{\mu\nu}\eta^{\mu\nu} \quad \Rightarrow \quad \eta_{\mu\nu}g^{\mu\nu} + h_{\mu\nu}g^{\mu\nu} = \eta_{\mu\nu}\eta^{\mu\nu}$$

Now, we can rewrite the second term as $h_{\mu\nu}g^{\mu\nu} = h^{\mu\nu}g_{\mu\nu}$. Here, the upper-index metric perturbation means $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$, because indices are raised with the Minkowski metric. We can insert $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ into this once more to get:

$$\begin{aligned}
&\eta_{\mu\nu}g^{\mu\nu} + h^{\mu\nu}g_{\mu\nu} = \eta_{\mu\nu}\eta^{\mu\nu} \\
&\Rightarrow \quad \eta_{\mu\nu}g^{\mu\nu} + h^{\mu\nu}(\eta_{\mu\nu} + h_{\mu\nu}) = \eta_{\mu\nu}\eta^{\mu\nu} \\
&\Rightarrow \quad \eta_{\mu\nu}g^{\mu\nu} + h^{\mu\nu}\eta_{\mu\nu} + \underbrace{h^{\mu\nu}h_{\mu\nu}}_{\approx 0} = \eta_{\mu\nu}\eta^{\mu\nu} \\
&\Rightarrow \quad \eta_{\mu\nu}g^{\mu\nu} = \eta_{\mu\nu}\left(\eta^{\mu\nu} - h^{\mu\nu}\right)
\end{aligned}$$

From this, we can identify the inverse metric as:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$$

So, the inverse metric picks up a minus sign but otherwise has the same kind of form as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The perturbation $h^{\mu\nu}$ with *upper indices* here is $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$.

With this, we can now calculate the Christoffel symbols from the general formula:

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2}g^{\lambda\alpha}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}) \\ &= \frac{1}{2}\left(\eta^{\lambda\alpha} - h^{\lambda\alpha}\right)(\partial_\mu(\eta_{\nu\alpha} + h_{\nu\alpha}) + \partial_\nu(\eta_{\mu\alpha} + h_{\mu\alpha}) - \partial_\alpha(\eta_{\mu\nu} + h_{\mu\nu})) \\ &= \frac{1}{2}\eta^{\lambda\alpha}(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu}) - \frac{1}{2}h^{\lambda\alpha}(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu})\end{aligned}$$

Here, we used the fact that derivatives of the Minkowski metric are zero because of our background coordinate choice (this is the place where assuming a *Cartesian* background really begins to simplify things). In the second term, we have terms of the form $h^{\lambda\alpha}\partial_\mu h_{\nu\alpha}$, which vanish in our linearized approximation. We therefore have:

$$\Gamma^{\lambda}_{\mu\nu} \approx \frac{1}{2}\eta^{\lambda\alpha}(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu})$$

So, the Christoffel symbols involve raising indices with the Minkowski metric (as expected) and derivatives calculated directly from the metric perturbation.

1.2. The Linearized Einstein Field Equations

Our goal here, ultimately, is to write down the Einstein field equations for linearized gravity and solve them. For that, we'll need the Einstein tensor, which is constructed out of the Riemann tensor, Ricci tensor and Ricci scalar. Let's work these out next.

We'll begin with the Riemann tensor, which involves both derivatives of Christoffel symbols and products of Christoffel symbols. These products of Christoffel symbols, which schematically have the form $\Gamma\Gamma \sim \eta\partial h\eta\partial h \sim (\partial h)^2$, are *non-linear* in the metric. These should vanish in our linearized approximation, which gives us for the Riemann tensor:

$$R^{\lambda}_{\mu\beta\nu} = \partial_\beta\Gamma^{\lambda}_{\mu\nu} - \partial_\nu\Gamma^{\lambda}_{\mu\beta} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\beta} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\lambda}_{\alpha\nu} \approx \partial_\beta\Gamma^{\lambda}_{\mu\nu} - \partial_\nu\Gamma^{\lambda}_{\mu\beta}$$

Let's plug in the Christoffel symbols we found earlier (the terms in red cancel each other):

$$
\begin{aligned}
R^{\lambda}_{\mu\beta\nu} &= \partial_\beta \left(\overbrace{\frac{1}{2}\eta^{\lambda\alpha}(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha} - \partial_\alpha h_{\mu\nu})}^{\Gamma^{\lambda}_{\mu\nu}} \right) - \partial_\nu \left(\overbrace{\frac{1}{2}\eta^{\lambda\alpha}(\partial_\mu h_{\beta\alpha} + \partial_\beta h_{\mu\alpha} - \partial_\alpha h_{\mu\beta})}^{\Gamma^{\lambda}_{\mu\beta}} \right) \\
&= \frac{1}{2}\eta^{\lambda\alpha}(\partial_\beta\partial_\mu h_{\nu\alpha} + \partial_\beta\partial_\nu h_{\mu\alpha} - \partial_\beta\partial_\alpha h_{\mu\nu} - \partial_\nu\partial_\mu h_{\beta\alpha} - \partial_\nu\partial_\beta h_{\mu\alpha} + \partial_\nu\partial_\alpha h_{\mu\beta}) \\
&= \frac{1}{2}\eta^{\lambda\alpha}(\partial_\beta\partial_\mu h_{\nu\alpha} - \partial_\beta\partial_\alpha h_{\mu\nu} - \partial_\nu\partial_\mu h_{\beta\alpha} + \partial_\nu\partial_\alpha h_{\mu\beta})
\end{aligned}
$$

This is our Riemann tensor. For the Ricci tensor, we need to contract the indices λ and β:

$$
\begin{aligned}
R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} &= \frac{1}{2}\eta^{\lambda\alpha}(\partial_\lambda\partial_\mu h_{\nu\alpha} - \partial_\lambda\partial_\alpha h_{\mu\nu} - \partial_\nu\partial_\mu h_{\lambda\alpha} + \partial_\nu\partial_\alpha h_{\mu\lambda}) \\
&= \frac{1}{2}\left(\eta^{\lambda\alpha}\partial_\lambda\partial_\mu h_{\nu\alpha} - \eta^{\lambda\alpha}\partial_\lambda\partial_\alpha h_{\mu\nu} - \partial_\nu\partial_\mu\left(\eta^{\lambda\alpha}h_{\lambda\alpha}\right) + \eta^{\lambda\alpha}\partial_\nu\partial_\alpha h_{\mu\lambda}\right)
\end{aligned}
$$

We've moved the Minkowski metric inside the derivatives in the third term, where we can define $\eta^{\lambda\alpha}h_{\lambda\alpha} = h$ (i.e. the trace of the metric perturbation - for the Minkowski metric, this is always $\eta^{\lambda\alpha}\eta_{\lambda\alpha} = 4$, but we don't know what it is for the metric perturbation). On the other terms, the Minkowski metric simply raises indices as $\eta^{\lambda\alpha}\partial_\lambda = \partial^\alpha$. We then have:

$$
\begin{aligned}
R_{\mu\nu} &= \frac{1}{2}\left(\eta^{\lambda\alpha}\partial_\lambda\partial_\mu h_{\nu\alpha} - \eta^{\lambda\alpha}\partial_\lambda\partial_\alpha h_{\mu\nu} - \partial_\nu\partial_\mu\left(\eta^{\lambda\alpha}h_{\lambda\alpha}\right) + \eta^{\lambda\alpha}\partial_\nu\partial_\alpha h_{\mu\lambda}\right) \\
&= \frac{1}{2}\left(\partial_\mu\partial^\alpha h_{\nu\alpha} - \partial^\alpha\partial_\alpha h_{\mu\nu} - \partial_\nu\partial_\mu h + \partial_\nu\partial^\lambda h_{\mu\lambda}\right) \\
&= \frac{1}{2}\left(\partial_\mu\partial^\alpha h_{\nu\alpha} + \partial_\nu\partial^\alpha h_{\mu\alpha} - \partial_\alpha\partial^\alpha h_{\mu\nu} - \partial_\nu\partial_\mu h\right)
\end{aligned}
$$

In the last step, we've relabeled the dummy index λ to an α and moved some of the terms around. There is no real content in this, other than make the formula a bit nicer-looking!

Next, let's do the Ricci scalar. For that, we need to contract $R_{\mu\nu}$ with the Minkowski metric:

$$\begin{aligned} R = \eta^{\mu\nu} R_{\mu\nu} &= \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu \partial^\alpha h_{\nu\alpha} + \eta^{\mu\nu}\partial_\nu \partial^\alpha h_{\mu\alpha} - \partial_\alpha \partial^\alpha \left(\eta^{\mu\nu} h_{\mu\nu}\right) - \eta^{\mu\nu}\partial_\nu \partial_\mu h\right) \\ &= \frac{1}{2}\left(\partial^\nu \partial^\alpha h_{\nu\alpha} + \partial^\mu \partial^\alpha h_{\mu\alpha} - \partial_\alpha \partial^\alpha h - \partial^\mu \partial_\mu h\right) \end{aligned}$$

We only summation indices in all of the terms here, which we can be relabel freely. So, relabeling $\nu \rightarrow \mu$ in the first term and $\mu \rightarrow \alpha$ in the third term, we can combine the terms:

$$R = \frac{1}{2}\left(\partial^\mu \partial^\alpha h_{\mu\alpha} + \partial^\mu \partial^\alpha h_{\mu\alpha} - \partial_\alpha \partial^\alpha h - \partial^\alpha \partial_\alpha h\right) = \partial^\mu \partial^\alpha h_{\mu\alpha} - \partial^\alpha \partial_\alpha h \overset{\mu \rightarrow \beta}{=} \partial^\beta \partial^\alpha h_{\beta\alpha} - \partial^\alpha \partial_\alpha h$$

Let's construct the Einstein tensor next. All we need to do is plug the Ricci tensor and Ricci scalar in $G_{\mu\nu} = R_{\mu\nu} - R\eta_{\mu\nu}/2$. We use $\eta_{\mu\nu}$ here instead of $g_{\mu\nu}$ as per the linearized framework (we would end up with non-linear terms if we used the full $g_{\mu\nu}$ here). This gives:

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2}\left(\partial_\mu \partial^\alpha h_{\nu\alpha} + \partial_\nu \partial^\alpha h_{\mu\alpha} - \partial_\alpha \partial^\alpha h_{\mu\nu} - \partial_\nu \partial_\mu h\right) - \frac{1}{2}\left(\partial^\beta \partial^\alpha h_{\alpha\beta} - \partial^\alpha \partial_\alpha h\right)\eta_{\mu\nu} \\ &= \frac{1}{2}\left(\partial_\mu \partial^\alpha h_{\nu\alpha} + \partial_\nu \partial^\alpha h_{\mu\alpha} - \eta_{\mu\nu}\partial^\beta \partial^\alpha h_{\alpha\beta} - \partial_\alpha \partial^\alpha h_{\mu\nu} - \partial_\mu \partial_\nu h + \eta_{\mu\nu}\partial^\alpha \partial_\alpha h\right) \end{aligned}$$

This is pretty much as far as we can simplify the Einstein tensor for now. We can now write down the Einstein field equations. For a general energy-momentum tensor $T_{\mu\nu}$, they are:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\Rightarrow \quad \boxed{\partial_\mu \partial^\alpha h_{\nu\alpha} - \partial_\mu \partial_\nu h + \partial_\nu \partial^\alpha h_{\mu\alpha} - \eta_{\mu\nu}\partial^\beta \partial^\alpha h_{\alpha\beta} - \partial^\alpha \partial_\alpha h_{\mu\nu} + \partial^\alpha \partial_\alpha \eta_{\mu\nu} h = \frac{16\pi G}{c^4} T_{\mu\nu}}$$

These are the **Einstein field equations for linearized gravity**. We will be able to simplify them a lot still by exploiting some of the freedom - via so-called *gauge transformations* - we have in defining the metric perturbation $h_{\mu\nu}$. In fact, we will be able to find the general solution to these equations for (almost) any energy-momentum tensor.

2. An Introduction To Gauge Transformations

The next topic we'll discuss are **gauge transformations** for linearized gravity. The idea is that writing the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ leaves some ambiguity in how the perturbation $h_{\mu\nu}$ is chosen with respect to the background $\eta_{\mu\nu}$. There turn out to be many physically *equivalent* perturbations, which are all related by something called *gauge transformations*.

Gauge transformations are at the heart of any application of linearized gravity, so we'll be using them throughout this lesson. I cover them in a more general context in my book *Field Theory For The Non-Physicist* if that's something you want to learn more about. What we do here with linearized gravity is just one example use case of **gauge field theories**.

2.1. What Are Gauge Transformations?

Gauge transformations, in the context of linearized gravity, are a special set of *coordinate transformations* that **leave the background metric $\eta_{\mu\nu}$ unchanged** but change the perturbation $h_{\mu\nu}$. That might sound weird at first since we would expect $\eta_{\mu\nu}$ -like the components of *any* tensor - to change under coordinate transformations.

The key here is that it's the full metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ that matters, and under any coordinate transformation, the components $g_{\mu\nu}$ do indeed transform like they should. However, because of linearity, certain transformations of $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ can be *recast* as gauge transformations of $h_{\mu\nu}$, while keeping the background $\eta_{\mu\nu}$ *fixed*. The full metric is transformed, but any terms resulting from that transformation can be "absorbed" into $h_{\mu\nu}$.

Let's develop this idea further now. We can start from the fact that under an arbitrary coordinate transformation, the components of the metric should transform as:

$$\overline{g}_{\mu\nu} = \Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu}g_{\alpha\beta} = \frac{\partial x^{\alpha}}{\partial \overline{x}^{\mu}}\frac{\partial x^{\beta}}{\partial \overline{x}^{\nu}}(\eta_{\alpha\beta} + h_{\alpha\beta}) = \frac{\partial x^{\alpha}}{\partial \overline{x}^{\mu}}\frac{\partial x^{\beta}}{\partial \overline{x}^{\nu}}\eta_{\alpha\beta} + \frac{\partial x^{\alpha}}{\partial \overline{x}^{\mu}}\frac{\partial x^{\beta}}{\partial \overline{x}^{\nu}}h_{\alpha\beta}$$

Here, $\Lambda^{\alpha}_{\mu} = \partial x^{\alpha} / \partial \overline{x}^{\mu}$ is the Jacobian matrix for the coordinate transformation from a set of coordinates x to some new set of coordinates, $\overline{x}$. The Jacobian matrix was covered in Lesson 2.

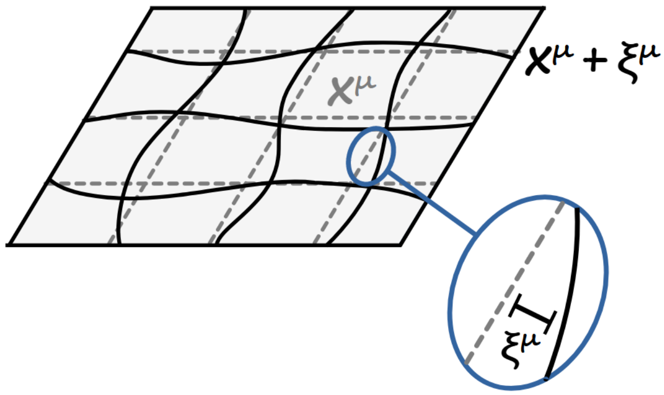
These two terms here describe how the background Minkowski metric $\eta_{\alpha\beta}$ and the perturbation $h_{\alpha\beta}$ transform under a completely general coordinate transformation:

$$\begin{cases} \overline{\eta}_{\mu\nu} = \dfrac{\partial x^\alpha}{\partial \overline{x}^\mu}\dfrac{\partial x^\beta}{\partial \overline{x}^\nu}\eta_{\alpha\beta} \\ \overline{h}_{\mu\nu} = \dfrac{\partial x^\alpha}{\partial \overline{x}^\mu}\dfrac{\partial x^\beta}{\partial \overline{x}^\nu}h_{\alpha\beta} \end{cases} \text{, and } \overline{g}_{\mu\nu} = \overline{\eta}_{\mu\nu} + \overline{h}_{\mu\nu}.$$

Gauge transformations are now the subset of transformations that **leave the background fixed**, so $\overline{\eta}_{\mu\nu} = \eta_{\mu\nu}$. These turn out to be transformations having the following form:

$$\boxed{\overline{x}^\mu = x^\mu + \xi^\mu}$$

This is a transformation that shifts each point x^μ by a *small* amount ξ^μ. Small, in this context, means the same it did for h, so anything of the form ξ^2, $(\partial\xi)^2$, $\xi\partial\xi$, or $h\xi$ can be neglected. The shift ξ^μ can vary from point to point, so $\xi^\mu \equiv \xi^\mu(x)$. This kind of coordinate transformation is essentially an arbitrary, but *infinitesimal* coordinate shift:



To understand what this coordinate transformation does, we'll need the Jacobian matrix. The "old" coordinates (*without* a bar) are given by $x^\mu = \overline{x}^\mu - \xi^\mu$, so the Jacobian matrix is:

$$\Lambda^\alpha_{\overline{\mu}} = \frac{\partial x^\alpha}{\partial \overline{x}^\mu} = \frac{\partial}{\partial \overline{x}^\mu}\left(\overline{x}^\alpha - \xi^\alpha\right) = \delta^\alpha_\mu - \frac{\partial \xi^\alpha}{\partial \overline{x}^\mu}$$

Here, we've used the fact that $\partial\overline{x}^\alpha / \partial\overline{x}^\mu$ equals 1 if $\alpha = \mu$ and zero otherwise: so, $\partial\overline{x}^\alpha / \partial\overline{x}^\mu = \delta^\alpha_\mu$.

We can write the derivative $\partial \xi^\alpha / \partial \bar{x}^\mu$ here by using the chain rule as:

$$\Lambda^{\alpha}_{\bar{\mu}} = \delta^{\alpha}_{\mu} - \frac{\partial \xi^\alpha}{\partial \bar{x}^\mu} = \delta^{\alpha}_{\mu} - \frac{\partial x^\nu}{\partial \bar{x}^\mu}\frac{\partial \xi^\alpha}{\partial x^\nu}$$

But $\partial x^\nu / \partial \bar{x}^\mu$ is just the Jacobian again, which we just found to be $\Lambda^{\nu}_{\bar{\mu}} = \delta^{\nu}_{\mu} - \partial \xi^\nu / \partial \bar{x}^\mu$. Then, using the approximation $(\partial \xi)^2 \approx 0$ (because ξ and its derivatives are small):

$$\Lambda^{\alpha}_{\bar{\mu}} = \delta^{\alpha}_{\mu} - \left(\delta^{\nu}_{\mu} - \frac{\partial \xi^\nu}{\partial \bar{x}^\mu}\right)\frac{\partial \xi^\alpha}{\partial x^\nu} = \delta^{\alpha}_{\mu} - \underbrace{\delta^{\nu}_{\mu}\frac{\partial \xi^\alpha}{\partial x^\nu}}_{=\partial \xi^\alpha/\partial x^\mu} + \underbrace{\frac{\partial \xi^\nu}{\partial \bar{x}^\mu}\frac{\partial \xi^\alpha}{\partial x^\nu}}_{\approx 0} \approx \delta^{\alpha}_{\mu} - \frac{\partial \xi^\alpha}{\partial x^\mu} \equiv \delta^{\alpha}_{\mu} - \partial_\mu \xi^\alpha$$

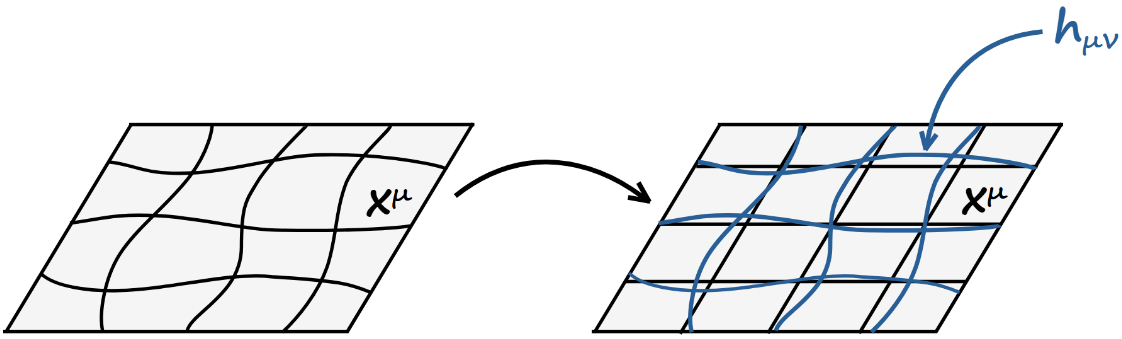
This result tells us that, in this context, there is no difference between taking derivatives with respect to the "new" coordinates $\bar{x}$ and the "original" coordinates x. We can now calculate how our metric tensor components transform as follows:

$$\begin{aligned}
\bar{g}_{\mu\nu} &= \Lambda^{\alpha}_{\bar{\mu}}\Lambda^{\beta}_{\bar{\nu}}\eta_{\alpha\beta} + \Lambda^{\alpha}_{\bar{\mu}}\Lambda^{\beta}_{\bar{\nu}}h_{\alpha\beta} \\
&= \left(\delta^{\alpha}_{\mu} - \partial_\mu \xi^\alpha\right)\left(\delta^{\beta}_{\nu} - \partial_\nu \xi^\beta\right)\eta_{\alpha\beta} + \left(\delta^{\alpha}_{\mu} - \partial_\mu \xi^\alpha\right)\left(\delta^{\beta}_{\nu} - \partial_\nu \xi^\beta\right)h_{\alpha\beta} \\
&= \underbrace{\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\eta_{\alpha\beta}}_{=\eta_{\mu\nu}} - \partial_\mu \xi^\alpha \underbrace{\delta^{\beta}_{\nu}\eta_{\alpha\beta}}_{=\eta_{\alpha\nu}} - \partial_\nu \xi^\beta \underbrace{\delta^{\alpha}_{\mu}\eta_{\alpha\beta}}_{=\eta_{\mu\beta}} + \underset{\approx 0}{\cancel{\partial_\mu \xi^\alpha \partial_\nu \xi^\beta \eta_{\alpha\beta}}} + \underbrace{\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}h_{\alpha\beta}}_{=h_{\mu\nu}} - \cancel{\partial_\mu \xi^\alpha \delta^{\beta}_{\nu} h_{\alpha\beta}} \\
&\quad - \cancel{\delta^{\alpha}_{\mu}\partial_\nu \xi^\beta h_{\alpha\beta}} + \cancel{\partial_\mu \xi^\alpha \partial_\nu \xi^\beta h_{\alpha\beta}} \\
&= \eta_{\mu\nu} - \partial_\mu \underbrace{\xi^\alpha \eta_{\alpha\nu}}_{=\xi_\nu} - \partial_\nu \underbrace{\xi^\beta \eta_{\mu\beta}}_{=\xi_\mu} + h_{\mu\nu} \\
&= \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu
\end{aligned}$$

Notice this contains only the *original* background Minkowski metric $\eta_{\mu\nu}$! The effect of the transformation are these $\partial\xi$-terms, which we can "absorb" into a transformed perturbation $\bar{h}_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$. We could then interpret the whole thing as a transformation of *only* $h_{\mu\nu}$, with the background coordinates themselves kept fixed:

$$\begin{cases} \bar{\eta}_{\mu\nu} = \eta_{\mu\nu} \\ \bar{h}_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \end{cases}$$

This is precisely what we mean by a **gauge transformation**: a coordinate transformation recast as a transformation of the "field" $h_{\mu\nu}$. So, the metric perturbation is now better thought of as an "external" tensor field on top of a fixed background spacetime:



This kind of "recasting" allows us to forget about coordinate transformations altogether and to think of these transformations as being done directly to the metric perturbation $h_{\mu\nu}$. Of course, the math is still the same regardless of how we choose to interpret it.

Gauge transformations in linearized gravity are direct transformations of the metric perturbation of the form $h_{\mu\nu} \rightarrow\ h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$, where ξ_μ is an arbitrary vector.

The really interesting thing about these gauge transformations is that they leave the curvature tensors *completely* unaffected. We can see this explicitly by looking at the Riemann tensor. The "gauge-transformed" Riemann tensor components $\overline{R}^{\lambda}_{\ \mu\beta\nu}$ are:

$$
\begin{aligned}
\overline{R}^{\lambda}_{\ \mu\beta\nu} &= \frac{1}{2}\eta^{\lambda\alpha}(\partial_\beta\partial_\mu\overline{h}_{\nu\alpha} - \partial_\beta\partial_\alpha\overline{h}_{\mu\nu} - \partial_\nu\partial_\mu\overline{h}_{\beta\alpha} + \partial_\nu\partial_\alpha\overline{h}_{\mu\beta}) \\
&= \frac{1}{2}\eta^{\lambda\alpha}\Big[\partial_\beta\partial_\mu(h_{\nu\alpha} - \partial_\nu\xi_\alpha - \partial_\alpha\xi_\nu) - \partial_\beta\partial_\alpha(h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu) \\
&\qquad - \partial_\nu\partial_\mu(h_{\beta\alpha} - \partial_\beta\xi_\alpha - \partial_\alpha\xi_\beta) + \partial_\nu\partial_\alpha(h_{\mu\beta} - \partial_\mu\xi_\beta - \partial_\beta\xi_\mu)\Big] \\
&= \frac{1}{2}\eta^{\lambda\alpha}\Big[\partial_\beta\partial_\mu h_{\nu\alpha} - \partial_\beta\partial_\mu\partial_\nu\xi_\alpha - \partial_\beta\partial_\mu\partial_\alpha\xi_\nu - \partial_\beta\partial_\alpha h_{\mu\nu} + \partial_\beta\partial_\alpha\partial_\mu\xi_\nu + \partial_\beta\partial_\alpha\partial_\nu\xi_\mu \\
&\qquad -\partial_\nu\partial_\mu h_{\beta\alpha} + \partial_\nu\partial_\mu\partial_\beta\xi_\alpha + \partial_\nu\partial_\mu\partial_\alpha\xi_\beta + \partial_\nu\partial_\alpha h_{\mu\beta} - \partial_\nu\partial_\alpha\partial_\mu\xi_\beta - \partial_\nu\partial_\alpha\partial_\beta\xi_\mu\Big] \\
&= \frac{1}{2}\eta^{\lambda\alpha}(\partial_\beta\partial_\mu h_{\nu\alpha} - \partial_\beta\partial_\alpha h_{\mu\nu} - \partial_\nu\partial_\mu h_{\beta\alpha} + \partial_\nu\partial_\alpha h_{\mu\beta}) \equiv R^{\lambda}_{\ \mu\beta\nu}
\end{aligned}
$$

We can see here that $\overline{R}^{\lambda}_{\mu\beta\nu} = R^{\lambda}_{\mu\beta\nu}$, so every component of the Riemann is **invariant** under a gauge transformation! A general coordinate transformation, while having no *physical* effect, would still change the components like $\overline{R}^{\lambda}_{\mu\beta\nu} = \Lambda^{\tau}_{\overline{\nu}}\Lambda^{\sigma}_{\overline{\beta}}\Lambda^{\kappa}_{\overline{\mu}}\Lambda^{\overline{\lambda}}_{\alpha}R^{\alpha}_{\kappa\sigma\tau}$. But a gauge transformation does not even do that, rather, it leaves them completely unchanged.

This shows directly that the choice of the perturbation $h_{\mu\nu}$, and how we identify it with the background spacetime, is *not* unique - there are many different $h_{\mu\nu}$'s that produce exactly the same curvature components. Because the Einstein field equations are also consequently invariant, they can only fix the perturbation $h_{\mu\nu}$ up to a *gauge*. We are free to choose this gauge however we like, a feature called **gauge symmetry**.

One consequence of this is that a perturbation of form of a *pure gauge* (meaning it can be written $h_{\mu\nu} = -\partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu}$) always has **zero curvature**. Curvature and gravity are therefore entirely contained in the *gauge-invariant* parts of the perturbation $h_{\mu\nu}$.

To summarize, we can recast linearized gravity as a gauge field theory of the metric perturbation $h_{\mu\nu}$. This allows us to keep the same *background* spacetime, but leaves us with freedom in choosing how $h_{\mu\nu}$ should be identified with respect to that background. All gauges still describe the same physics, some just might be more convenient in practice.

2.2. The Lorenz (Harmonic) Gauge

In this lesson, we'll mainly work with the so-called the *Lorenz gauge*. The Lorenz gauge is perhaps the most widely used gauge in linearized gravity and it works well for describing both the *Newtonian limit* and the physics of *gravitational waves*. It also works for analyzing so-called *post-Newtonian* effects, which we'll get a glimpse of later as well.

The Lorenz gauge is usually applied, not directly to $h_{\mu\nu}$, but to its **trace-reversed** version, which we define as the following combination of objects:

$$\boxed{h^{\mathrm{TR}}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h}$$

Trace-reversed here means that $\mathrm{Tr}\left(h^{\mathrm{TR}}\right) = \eta^{\mu\nu}h^{\mathrm{TR}}_{\mu\nu} = -h$ *- so, a sign reversal of the trace.*

The **Lorenz gauge** now uses the gauge symmetry discussed earlier to set the (four-)*divergence* of this trace-reversed perturbation to zero:

The Lorenz gauge

$$\partial^{\mu} h^{\text{TR}}_{\mu\nu} = 0$$

Here, we have a summation over the index μ, so schematically, this would read:

$$\partial^0 h^{\text{TR}}_{0\nu} + \partial^1 h^{\text{TR}}_{1\nu} + \partial^2 h^{\text{TR}}_{2\nu} + \partial^3 h^{\text{TR}}_{3\nu} \quad \Rightarrow \quad \text{"}\nabla \cdot h^{\text{TR}}_{i\nu}\text{"} - \frac{1}{c}\partial_t h^{\text{TR}}_{0\nu} = 0$$

If you're familiar with electrodynamics, this is the "linearized gravity version" of the Lorenz gauge $\nabla \cdot \vec{A} + \partial_t \phi / c = 0$ used there. The difference here is that the Lorenz gauge is actually **four conditions**, one for each value of the index $\nu = \{0, 1, 2, 3\} \equiv \{0, i\}$. Degree-of-freedom-wise, imposing the Lorenz gauge reduces the number of independent components of the metric perturbation $h_{\mu\nu}$ from 10 to 6.

One question this raises is, how do we know this type of gauge is something we're allowed to impose? The way to check if a gauge is valid is to check that it can really be written in the form of a gauge transformation. This is analyzed in a bit more detail below.

What Does It Mean To Impose The Lorenz Gauge?

First, let's remind ourselves of the general form of our gauge transformation. A gauge transformation generally takes us from $h_{\mu\nu}$ to a new perturbation $\overline{h}_{\mu\nu}$ of the form:

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu}$$

In terms of degrees of freedom, a gauge transformation corresponds to choosing a gauge parameter ξ_{μ} with **four** components. The Lorenz gauge imposes exactly four such conditions on the perturbation, so it works out degree-of-freedom-wise at least.

However, just imposing any four conditions does not automatically make for a valid gauge. We also have to check that the transformation can really be written in terms of some four-vector ξ_μ. Explicitly, the gauge conditions are always imposed on $\overline{h}_{\mu\nu}$, the *gauge-transformed* perturbation (though we usually drop the bar notation afterwards).

For the Lorenz gauge, we need to first know how the trace-reversed metric transforms under gauge transformations. Our "original" $h_{\mu\nu}$ as well as its trace h transform as:

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$
$$\overline{h} = \eta^{\mu\nu}\overline{h}_{\mu\nu} = \eta^{\mu\nu}(h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu) = \underbrace{\eta^{\mu\nu}h_{\mu\nu}}_{=h} - \underbrace{\eta^{\mu\nu}\partial_\mu}_{=\partial^\nu} \xi_\nu - \underbrace{\eta^{\mu\nu}\partial_\nu}_{=\partial^\mu} \xi_\mu = h - 2\partial^\alpha \xi_\alpha$$

To get to the last equality, we replaced both summation indices $\nu, \mu \to \alpha$ and combined the terms.

The trace-reversed perturbation would itself then transform as:

$$\begin{aligned}\overline{h}^{\mathrm{TR}}_{\mu\nu} &= \overline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overline{h} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu - \frac{1}{2}\eta_{\mu\nu}\left(h - 2\partial^\alpha \xi_\alpha\right) \\ &= \underbrace{h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h}_{h^{\mathrm{TR}}_{\mu\nu}} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial^\alpha \xi_\alpha = h^{\mathrm{TR}}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial^\alpha \xi_\alpha\end{aligned}$$

The Lorenz gauge condition would now set $\partial^\mu \overline{h}^{\mathrm{TR}}_{\mu\nu} = 0$, which requires:

$$\begin{aligned}&\partial^\mu \overline{h}^{\mathrm{TR}}_{\mu\nu} = 0 \\ \Rightarrow\quad &\partial^\mu\left(h^{\mathrm{TR}}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial^\alpha \xi_\alpha\right) = 0 \\ \Rightarrow\quad &\partial^\mu h^{\mathrm{TR}}_{\mu\nu} - \partial^\mu \partial_\mu \xi_\nu - \partial_\nu \partial^\mu \xi_\mu + \underbrace{\eta_{\mu\nu}\partial^\mu}_{=\partial_\nu} \underbrace{\partial^\alpha \xi_\alpha}_{\alpha\to\mu} = 0 \\ \Rightarrow\quad &\partial^\mu h^{\mathrm{TR}}_{\mu\nu} - \partial^\mu \partial_\mu \xi_\nu - \partial_\nu \partial^\mu \xi_\mu + \partial_\nu \partial^\mu \xi_\mu = 0 \\ \Rightarrow\quad &\partial^\mu \partial_\mu \xi_\nu = \partial^\mu h^{\mathrm{TR}}_{\mu\nu}\end{aligned}$$

These are partial differential equations we need to solve for the gauge parameter $\xi_\nu = (\xi_0, \xi_i)$ that puts us into the Lorenz gauge. The idea here is that we have some "original" metric perturbation $h_{\mu\nu}$, which defines $h^{\mathrm{TR}}_{\mu\nu}$. Then, imposing the Lorenz gauge on this metric amounts to solving the above partial differential equations.

We plug in our original $h_{\mu\nu}^{\text{TR}}$, and if we *can* find a valid solution for ξ_ν, we can indeed impose the Lorenz gauge by using this ξ_ν to do the transformation $h_{\mu\nu}^{\text{TR}} \rightarrow \bar{h}_{\mu\nu}^{\text{TR}}$.

But *can* we actually solve these differential equations? It turns out that *we can*. These equations have a generally well-known form of a wave equation with the divergence of our original $h_{\mu\nu}^{\text{TR}}$ as a "source term", and we generally know that such an equation always has a solution (under suitable boundary conditions). The important part is that simply knowing a solution exists is enough to say that **the Lorenz gauge can always be imposed**. We don't even need to solve for ξ_ν, it's enough to note that *some* solution always exists - when it does, we can use gauge freedom to set $\partial^\mu \bar{h}_{\mu\nu}^{\text{TR}} = 0$.

After imposing the Lorenz gauge, we usually drop the "gauge-transformed" bar-notation and just write $h_{\mu\nu}^{\text{TR}}$ instead of $\bar{h}_{\mu\nu}^{\text{TR}}$. We just need to be careful then that $h_{\mu\nu}^{\text{TR}}$ (or $h_{\mu\nu}$) is now specifically defined *in* the Lorenz gauge, so it satisfies $\partial^\mu h_{\mu\nu}^{\text{TR}} = 0$).

Next, let's look at the Einstein field equations in the Lorenz gauge, as this is where the usefulness of this gauge becomes clear. Recall the general linearized field equations:

$$\partial_\mu \partial^\alpha h_{\nu\alpha} - \partial_\mu \partial_\nu h + \partial_\nu \partial^\alpha h_{\mu\alpha} - \eta_{\mu\nu} \partial^\beta \partial^\alpha h_{\alpha\beta} - \partial^\alpha \partial_\alpha h_{\mu\nu} + \partial^\alpha \partial_\alpha \eta_{\mu\nu} h = \frac{16\pi G}{c^4} T_{\mu\nu}$$

By writing $\partial_\mu \partial_\nu h = (\partial_\mu \partial_\nu h + \partial_\mu \partial_\nu h)/2$, we can rewrite the terms with ∂_μ- and ∂_ν-derivatives (by writing them as $\partial_\mu = \eta_{\mu\alpha} \partial^\alpha$ and $\partial_\nu = \eta_{\alpha\nu} \partial^\alpha$) in terms of the trace-reversed perturbation and apply the Lorenz gauge to them:

$$\partial_\mu \partial^\alpha h_{\nu\alpha} - \frac{1}{2} \partial_\mu \partial_\nu h - \frac{1}{2} \partial_\mu \partial_\nu h + \partial_\nu \partial^\alpha h_{\mu\alpha} - \eta_{\mu\nu} \partial^\beta \partial^\alpha h_{\alpha\beta} - \partial^\alpha \partial_\alpha h_{\mu\nu} + \partial^\alpha \partial_\alpha \eta_{\mu\nu} h = \frac{16\pi G}{c^4} T_{\mu\nu}$$

$$\Rightarrow \ \partial_\mu \partial^\alpha \underbrace{\left(h_{\nu\alpha} - \frac{1}{2} \eta_{\alpha\nu} h \right)}_{h_{\nu\alpha}^{\text{TR}}} + \partial_\nu \partial^\alpha \underbrace{\left(h_{\mu\alpha} - \frac{1}{2} \eta_{\mu\alpha} h \right)}_{h_{\mu\alpha}^{\text{TR}}} - \eta_{\mu\nu} \partial^\beta \partial^\alpha h_{\alpha\beta} - \partial^\alpha \partial_\alpha h_{\mu\nu} + \partial^\alpha \partial_\alpha \eta_{\mu\nu} h = \frac{16\pi G}{c^4} T_{\mu\nu}$$

$$\Rightarrow \ \partial_\mu \partial^\alpha h_{\nu\alpha}^{\text{TR}} + \partial_\nu \partial^\alpha h_{\mu\alpha}^{\text{TR}} - \eta_{\mu\nu} \partial^\beta \partial^\alpha h_{\alpha\beta} - \partial^\alpha \partial_\alpha h_{\mu\nu} + \partial^\alpha \partial_\alpha \eta_{\mu\nu} h = \frac{16\pi G}{c^4} T_{\mu\nu}$$

$$\Rightarrow \ -\eta_{\mu\nu} \partial^\beta \partial^\alpha h_{\alpha\beta} - \partial^\alpha \partial_\alpha h_{\mu\nu} + \partial^\alpha \partial_\alpha \eta_{\mu\nu} h = \frac{16\pi G}{c^4} T_{\mu\nu}$$

There is still another simplification we can make. The Lorenz gauge, when written out, allows us to write the four-divergence of $h_{\alpha\beta}$ in the following form:

$$\partial^\beta\left(h_{\alpha\beta}-\frac{1}{2}\eta_{\alpha\beta}h\right)=0$$

$$\Rightarrow\quad \partial^\beta h_{\alpha\beta}-\frac{1}{2}\underbrace{\eta_{\alpha\beta}\partial^\beta}_{=\partial_\alpha}h=0$$

$$\Rightarrow\quad \partial^\beta h_{\alpha\beta}=\frac{1}{2}\partial_\alpha h$$

The Einstein field equations then take the following, much much simpler form:

$$-\eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta}-\partial^\alpha\partial_\alpha h_{\mu\nu}+\partial^\alpha\partial_\alpha\eta_{\mu\nu}h=\frac{16\pi G}{c^4}T_{\mu\nu}$$

$$\Rightarrow\quad -\frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial_\alpha h-\partial^\alpha\partial_\alpha h_{\mu\nu}+\partial^\alpha\partial_\alpha\eta_{\mu\nu}h=\frac{16\pi G}{c^4}T_{\mu\nu}$$

$$\Rightarrow\quad -\partial^\alpha\partial_\alpha\underbrace{\left(h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h\right)}_{h^{\mathrm{TR}}_{\mu\nu}}=\frac{16\pi G}{c^4}T_{\mu\nu}$$

$$\Rightarrow\quad \boxed{\partial^\alpha\partial_\alpha h^{\mathrm{TR}}_{\mu\nu}=-\frac{16\pi G}{c^4}T_{\mu\nu}}$$

These are the full **Einstein field equations in the Lorenz gauge**. Notice how much we were able to simplify them with just a gauge choice? These equations still contain *all* of the same physics as the "original" ones - a gauge choice doesn't change that. But what the Lorenz gauge does is, it allows us to see the *nature* of these field equations more clearly. If we write out the sum over α and use $\partial_\alpha=(\partial/\partial t/c,\partial_i)$, we would get:

$$\overbrace{\partial^0}^{-\partial_0}\partial_0 h^{\mathrm{TR}}_{\mu\nu}+\overbrace{\partial^i\partial_i}^{\nabla^2}h^{\mathrm{TR}}_{\mu\nu}=-\frac{16\pi G}{c^4}T_{\mu\nu}\quad\Rightarrow\quad\boxed{\nabla^2 h^{\mathrm{TR}}_{\mu\nu}-\frac{1}{c^2}\frac{\partial^2 h^{\mathrm{TR}}_{\mu\nu}}{\partial t^2}=-\frac{16\pi G}{c^4}T_{\mu\nu}}$$

This is a **wave equation** for the trace-reversed perturbation. It implies that wave-like solutions for $h_{\mu\nu}^{\mathrm{TR}}$ might - or should - be possible. The Lorenz gauge turns the Einstein field equations into wave equations and pretty much directly suggests the existence of *gravitational waves*. Again, I should stress here that this "wave-like" nature was always there in the original linearized field equations, the Lorenz gauge just made it obvious.

We'll postpone our discussion of gravitational waves for a bit later, after we've looked at the Newtonian limit first. These field equations, even though they seem to describe waves, actually reduce to Newtonian gravity when we take their stationary, non-relativistic limit.

3. The Newtonian Limit

At this point, we've covered the basics of gauge transformations and seen how the *Lorenz gauge* turns the linearized Einstein field equations into wave equations for out metric perturbation. We are now ready to dive into applications of linearized gravity, the first one of which is the **Newtonian limit**. The Newtonian limit is the **non-relativistic approximation** of general relativity, which expect should reproduce Newtonian gravity.

The non-relativistic limit assumes all speeds are much below the speed of light, so that gravitational effects appear essentially *instantaneously* everywhere. This is the Newtonian "action at a distance", which we'll see can be recovered from the Einstein field equations.

The plan for our analysis will go as follows. First, we'll write down and solve the field equations in the Newtonian, non-relativistic case. We'll see how Poisson's equation drops out from the field equations and what kind of solution for the metric perturbation we find.

After that, we'll extend this framework to describe gravity around a rotating mass, which results in the so-called *Lense-Thirring metric*. The Lense-Thirring metric describes a "nearly-Newtonian" gravitational field but with some **relativistic effects** like *frame dragging*.

3.1. The Newtonian Field Equations

Our goal is to now solve the Lorenz gauge field equations in the Newtonian limit. We can begin by first writing them out component-wise, which gives us three equations, the 00-, $0i$- and ij-equations (where the Latin index $i = \{1, 2, 3\}$ represents spatial components):

$$\partial^\alpha \partial_\alpha h_{\mu\nu}^{\mathrm{TR}} = -\frac{16\pi G}{c^4} T_{\mu\nu} \quad \Rightarrow \quad \begin{cases} \partial^\alpha \partial_\alpha h_{00}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^4} T_{00} \\ \partial^\alpha \partial_\alpha h_{0i}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^4} T_{0i} \\ \partial^\alpha \partial_\alpha h_{ij}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^4} T_{ij} \end{cases}$$

Our gauge freedom is already used, so the only way to make further simplifications is via *physical* assumptions. The **Newtonian limit** is specifically based on the assumption that our spacetime is **stationary** (time-independent), so $\partial_0 h_{\mu\nu} \approx 0$. This doesn't mean that a Newtonian gravitational field strictly cannot depend on time, it just has to vary *slowly* enough in time compared to space. Recall how the four-derivatives are defined:

$$\partial_\alpha h_{\mu\nu} = \begin{pmatrix} \partial_0 h_{\mu\nu} & \partial_i h_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{c}\dfrac{\partial h_{\mu\nu}}{\partial t} & \dfrac{\partial h_{\mu\nu}}{\partial x^i} \end{pmatrix}$$

All derivatives of $h_{\mu\nu}$ are already small, so the ∂_0-derivatives are *extra* small because they are divided by a factor of $c \approx 3 \times 10^8$. The assumption $\partial_0 h_{\mu\nu} \approx 0$ should be reasonable then, in which case the ∂_α-derivatives in the field equations all turn into ∂_i-derivatives:

$$\begin{cases} \partial^\alpha \partial_\alpha h_{00}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^4} T_{00} \\ \partial^\alpha \partial_\alpha h_{0i}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^4} T_{0i} \\ \partial^\alpha \partial_\alpha h_{ij}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^4} T_{ij} \end{cases} \quad \Rightarrow \quad \begin{cases} \nabla^2 h_{00}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^4} T_{00} \\ \nabla^2 h_{0i}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^4} T_{0i} \\ \nabla^2 h_{ij}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^4} T_{ij} \end{cases}$$

Reminder: in Cartesian coordinates, the Laplacian has the form $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 \equiv \partial^i \partial_i$.

These are the general, **stationary Einstein field equations** in the Lorenz gauge. Each equation has the form of a Poisson's equation with T_{00}, T_{0i} and T_{ij} as source terms.

To solve them and recover the Newtonian limit, we should specify an energy-momentum tensor $T_{\mu\nu}$. For this purpose, we'll assume an *idealized* matter distribution, a **perfect fluid** to be specific. The energy-momentum tensor of a perfect fluid has the general form (discussed in Lesson 5), defined in terms of mass density ρ and pressure p:

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu + p\eta_{\mu\nu}$$

In the slow-velocity limit (meaning the Lorentz factors $\gamma \approx 1$), these four-velocities can be written as $u_\mu = \begin{pmatrix} -c & v_i \end{pmatrix}$, with v_i the components of 3-velocity (the minus sign comes from lowering indices, $u_0 = \eta_{0\mu} u^\mu = -u^0 = -c$). The explicit components are then:

$$T_{00} = \left(\rho + \frac{p}{c^2}\right) u_0 u_0 + p\eta_{00} = \left(\rho + \frac{p}{c^2}\right) c^2 - p = \rho c^2$$

$$T_{0i} = \left(\rho + \frac{p}{c^2}\right) u_0 u_i + p\eta_{0i} = -\left(\rho + \frac{p}{c^2}\right) c v_i$$

$$T_{ij} = \left(\rho + \frac{p}{c^2}\right) u_i u_j + p\eta_{ij} = \left(\rho + \frac{p}{c^2}\right) v_i v_j + p\delta_{ij}$$

Here's where we make our second physical assumption: Newtonian gravitational fields are primarily dominated by **mass** and gravitational effects from pressure are negligible. Mathematically, this means that $p/\rho c^2 \approx 0$. The $0i$- and ij-components are then:

$$T_{0i} = -\left(1 + \frac{p}{\rho c^2}\right) \rho c v_i \approx -\rho c v_i$$

$$T_{ij} = \left(1 + \frac{p}{\rho c^2}\right) \rho v_i v_j + p\delta_{ij} \approx \rho v_i v_j + p\delta_{ij}$$

Because the velocity components v_i are small, any resulting terms like v_i / c^3 or $v_i v_j / c^4$ we may find in the Einstein field equations are going to be *tiny*.

In the non-relativistic limit, in fact, the right-hand sides of the $0i$- and the ij-field equations approximate to zero due to this slow-velocity approximation (and $p/c^4 \approx 0$):

$$\begin{cases} \nabla^2 h_{00}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^2}\rho \\ \nabla^2 h_{0i}^{\mathrm{TR}} = \dfrac{16\pi G}{c^3}\rho v_i \\ \nabla^2 h_{ij}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^4}\rho v_i v_j - \dfrac{16\pi G}{c^4} p\delta_{ij} \end{cases} \Rightarrow \begin{cases} \nabla^2 h_{00}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^2}\rho \\ \nabla^2 h_{0i}^{\mathrm{TR}} \approx 0 \\ \nabla^2 h_{ij}^{\mathrm{TR}} \approx 0 \end{cases}$$

These are our **Newtonian field equations** (again in the Lorenz gauge). The $0i$- and ij-equations actually have very simple solutions: both have the form of **Laplace's equations**, which are generally solved by *harmonic functions*. If we additionally assume our gravitational field should go to zero at infinity (which it should physically!), the only possible such harmonic functions are precisely $h_{0i}^{\mathrm{TR}} = h_{ij}^{\mathrm{TR}} = 0$.

About harmonic functions: the solutions $h_{0i}^{\mathrm{TR}} = h_{ij}^{\mathrm{TR}} = 0$ actually come from just one property of harmonic functions: the minimum and maximum of any harmonic function can only occur at the *boundary* of the domain they are defined in. In our case, if Laplace's equations hold everywhere, then this domain is all of 3D space, so h_{0i}^{TR} and h_{ij}^{TR} must have their minimum and maximum at infinity. If both are also required to go to zero at infinity (a gravitational field should get *infinitely* weak if we go infinitely far from the source), the *extrema* of both must then be zero. The only functions with a minimum *and* maximum both being zero are exactly $h_{0i}^{\mathrm{TR}} = h_{ij}^{\mathrm{TR}} = 0$.

So, the unique solutions to the $0i$- and ij-equations which decay to zero at infinity are $h_{0i}^{\mathrm{TR}} = 0$ and $h_{ij}^{\mathrm{TR}} = 0$ (though, this is only true if $T_{0i} = 0$ truly everywhere). To solve the 00-equation, we should think about what we are actually trying to do here - recover Newtonian gravity. In Newtonian gravity, gravitational fields can be described by a Newtonian gravitational potential ϕ that satisfies **Poisson's equation**:

$$\nabla^2 \phi = 4\pi G\rho$$

If we want our solution for the metric to match this, we should have our 00-equation reduce to exactly this. This then relates h_{00}^{TR} to the Newtonian gravitational potential:

$$\nabla^2 h_{00}^{\mathrm{TR}} = -\frac{16\pi G}{c^2}\rho \quad \Rightarrow \quad h_{00}^{\mathrm{TR}} = -\frac{4\phi}{c^2}$$

The most general definition of the Newtonian gravitational potential is $\phi = -\iiint \frac{G\rho(x')}{|\vec{x}-\vec{x}'|} d^3x'$.

With this, we've essentially solved the field equations for the Newtonian limit: the solutions are $h_{00}^{\mathrm{TR}} = -4\phi/c^2$ and $h_{0i}^{\mathrm{TR}} = h_{ij}^{\mathrm{TR}} = 0$. But we are not quite done yet - these are components of the *trace-reversed* perturbation, which was an intermediary definition we made. To get the "original" metric perturbation $h_{\mu\nu}$, we can use the definition $h_{\mu\nu}^{\mathrm{TR}} = h_{\mu\nu} - \eta_{\mu\nu} h/2$. For the spatial components h_{ij}, this gives:

$$\underbrace{h_{ij}^{\mathrm{TR}}}_{\equiv 0} = h_{ij} - \frac{1}{2}\underbrace{\eta_{ij}}_{\delta_{ij}} h \quad \Rightarrow \quad h_{ij} = \frac{1}{2}\delta_{ij} h$$

If the spatial components are given by these, then the trace $h = \eta^{\mu\nu}h_{\mu\nu}$ is simply:

$$h = \underbrace{\eta^{00}}_{-1} h_{00} + \underbrace{\eta^{ij}}_{\delta^{ij}} h_{ij} = -h_{00} + \frac{1}{2}\underbrace{\delta^{ij}\delta_{ij}}_{=3} h = -h_{00} + \frac{3}{2}h \quad \Rightarrow \quad h = 2h_{00}$$

Using this, we then get for the 00-components:

$$\underbrace{h_{00}^{\mathrm{TR}}}_{-4\phi/c^2} = h_{00} - \frac{1}{2}\underbrace{\eta_{00}}_{-1} h = h_{00} + \frac{1}{2} \times 2h_{00} = 2h_{00} \quad \Rightarrow \quad h_{00} = -\frac{2\phi}{c^2}$$

For the $0i$-components, we have just $h_{0i} = h_{0i}^{\mathrm{TR}} = 0$. Our full solution is therefore:

$$\begin{cases} h_{00} = -\dfrac{2\phi}{c^2} \\ h_{0i} = 0 \\ h_{ij} = \dfrac{1}{2}\delta_{ij} h = h_{00}\delta_{ij} = -\dfrac{2\phi}{c^2}\delta_{ij} \end{cases}$$

We've now completely solved for the metric in the Newtonian limit! Collecting our results, we could write the full metric tensor in a 2×2 block matrix (the ij-block being 3×3):

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \begin{pmatrix} -1 + h_{00} & h_{0i} \\ h_{0i} & \delta_{ij} + h_{ij} \end{pmatrix} = \begin{pmatrix} -1 - \dfrac{2\phi}{c^2} & 0 \\ 0 & \left(1 - \dfrac{2\phi}{c^2}\right)\delta_{ij} \end{pmatrix}$$

Another quite common way to write this is in terms of the **line element**:

$$\boxed{ds^2 = -\left(1 + \frac{2\phi}{c^2}\right)c^2 dt^2 + \left(1 - \frac{2\phi}{c^2}\right)\delta_{ij} dx^i dx^j}$$

One useful special case of this is the spacetime around a *spherically symmetric* mass distribution, which has a Newtonian gravitational potential of $\phi(r) = -GM/r$, where r is the distance from the center of the mass. The line element would take the form:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 + \frac{2GM}{c^2 r}\right)\delta_{ij} dx^i dx^j$$

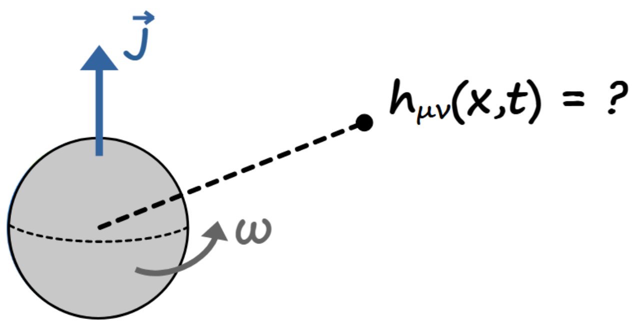
So, this is how Newtonian gravity, or the Newtonian limit, arises from general relativity - it all comes from the framework of *linearized gravity*, which we described here using the Lorenz gauge. Linearized gravity itself is much more general than just the Newtonian limit, but we recover Newtonian gravity specifically via the following assumptions:

- **Mass density** provides the dominant contribution to gravity.
- **Velocities are small** (non-relativistic), so our spacetime is approximately **stationary**.

If we *don't make* these assumptions, then we are not in the Newtonian limit anymore (though we are still in linearized gravity). Lifting some of these assumptions gives us what are often called *post-Newtonian* effects, which we'll take a brief look at next.

3.2. Post-Newtonian Effects: Lense-Thirring Metric

In the previous section, we saw how mass density provides the most important contribution to gravity in the Newtonian limit. However, other forms of energy gravitate too, they are just negligible in the Newtonian limit. In this section, we'll see how $T_{0i} \neq 0$ leads to an interesting, purely relativistic effect called **frame dragging** sourced by *angular momentum*. At the end, we'll get to the **Lense-Thirring metric**, which describes the spacetime around a weakly rotating mass (it's also the weak-field limit of the Kerr metric!).



Our plan here is to assume a **stationary** spacetime still, so $\partial_0 h_{\mu\nu} \approx 0$. This is valid in the slow-velocity limit, which means our mass must be rotating slowly. We are still also working in the **Lorenz gauge**, which is valid for any linearized spacetime.

For the energy-momentum tensor, we'll still assume a perfect fluid with mass density dominating over pressure (the approximation $p / \rho c^2 \approx 0$). Its components themselves are then exactly the same as before: $T_{00} = \rho c^2$, $T_{0i} \approx -\rho c v_i$ and $T_{ij} \approx \rho v_i v_j + p\delta_{ij}$.

The velocity components of our source can consist of a linear velocity v_i^{lin} and a rotational velocity of the form $v_i^{\text{rot}} = (\vec{\omega} \times \vec{r})_i$. If our matter configuration were some extended body spinning with angular velocity $\vec{\omega}$ around an axis, the total velocity of a given *point* on this body would be $v_i = v_i^{\text{lin}} + v_i^{\text{rot}}$. For the Lense-Thirring spacetime, we assume that the source is only *rotating* but not moving linearly, so that $v_i = v_i^{\text{rot}} = (\vec{\omega} \times \vec{r})_i$.

In the field equations, we will keep the source term of the 00-equation and now also the first relativistic correction in the $0i$-equations ($\propto \rho v_i / c^3$). We'll still neglect the squared velocity term and the pressure term in the ij-equations, so $v_i v_j / c^4 \approx 0$ and $p / c^4 \approx 0$.

The linearized Einstein's equations will then take the following form, using $v_i = (\vec{\omega} \times \vec{r})_i$:

$$\begin{cases} \nabla^2 h_{00}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^2}\rho \\ \nabla^2 h_{0i}^{\mathrm{TR}} = \dfrac{16\pi G}{c^3}\rho v_i \\ \nabla^2 h_{ij}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^4}\rho v_i v_j - \dfrac{16\pi G}{c^4} p\delta_{ij} \end{cases} \Rightarrow \boxed{\begin{cases} \nabla^2 h_{00}^{\mathrm{TR}} = -\dfrac{16\pi G}{c^2}\rho \\ \nabla^2 h_{0i}^{\mathrm{TR}} = \dfrac{16\pi G}{c^3}\rho(\vec{\omega} \times \vec{r})_i \\ \nabla^2 h_{ij}^{\mathrm{TR}} \approx 0 \end{cases}}$$

These are the field equations we should solve to find out how the rotation of the central mass ($\vec{\omega} \neq 0$) affects our linearized spacetime. These equations hold for any *weakly* gravitating and *slowly rotating* source, which could be for example Earth.

Now, the 00- and ij-equations here have exactly the same forms as before, so we would expect them to have the same solutions. They indeed do, so we will have just like before:

$$\begin{cases} h_{00}^{\mathrm{TR}} = -\dfrac{4\phi}{c^2} \\ h_{ij}^{\mathrm{TR}} = 0 \end{cases} \Rightarrow \begin{cases} h_{00} = -\dfrac{2\phi}{c^2} \\ h_{ij} = -\dfrac{2\phi}{c^2}\delta_{ij} \end{cases}$$

The only thing that has changed is the $0i$-equation, which will now give us a non-zero h_{0i} as a result. For simplicity, we can assume that our matter configuration is a **uniform sphere** (so its density ρ is constant) rotating with constant **angular momentum** $\vec{J}$. The solution *outside* the mass then turns out to be (you'll find the derivation of this below):

$$\boxed{h_{0i} = h_{0i}^{\mathrm{TR}} = -\frac{2G}{c^3 r^3}(\vec{J} \times \vec{r})_i}$$

Solution of The Lense-Thirring Field Equation

Our goal here is to solve the $0i$-component of the linearized field equations for the perturbation components h_{0i}. The field equations, however, are for h_{0i}^{TR} (the trace-reversed version of the perturbation) but because $\eta_{0i} = 0$ we have in this case that $h_{0i}^{\text{TR}} = h_{0i} - h\eta_{0i}/2 = h_{0i}$. We can therefore replace the trace-reversed components with the "original" $0i$-components, so the equation we need to solve becomes:

$$\nabla^2 h_{0i} = \frac{16\pi G\rho}{c^3}(\vec{\omega} \times \vec{r})_i$$

To solve this, we need to know what our rotating source actually looks like. We'll assume here that it is a (uniform) rotating sphere of radius R. A reasonable ansatz for the solution, based on linearity, would be that $h_{0i} \propto T_{0i} \propto (\vec{\omega} \times \vec{r})_i$. All the directional dependence is contained in the factor $\vec{\omega} \times \vec{r}$ but we don't know how the solution should depend on r. We should therefore also include a function $f(r)$ in our ansatz:

$$h_{0i} = f(r)(\vec{\omega} \times \vec{r})_i$$

This is just a *guess* at what the solution might look like. We now need to verify it actually works and solve for $f(r)$. To do so, we'll plug it into the field equation, and using the fact that $\vec{\omega}$ is just a *constant* vector, we can bring it outside the Laplacian:

$$\nabla^2(f(r)(\vec{\omega} \times \vec{r})_i) = \frac{16\pi G\rho}{c^3}(\vec{\omega} \times \vec{r})_i \quad \Rightarrow \quad \left(\vec{\omega} \times \nabla^2(f(r)\vec{r})\right)_i = \frac{16\pi G\rho}{c^3}(\vec{\omega} \times \vec{r})_i$$

Now, there exists a general formula for the Laplacian of a "radial function times $\vec{r}$". The formula says that the above equation can be written in the form:

$$\left(\vec{\omega} \times \underbrace{\left(\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df(r)}{dr}\right) + \frac{2}{r}\frac{df(r)}{dr}\right)\vec{r}}_{=\nabla^2(f(r)\vec{r}))}\right)_i = \frac{16\pi G\rho}{c^3}(\vec{\omega} \times \vec{r})_i$$

$$\Rightarrow \quad \left(\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df(r)}{dr}\right) + \frac{2}{r}\frac{df(r)}{dr}\right)(\vec{\omega}\times\vec{r})_i = \frac{16\pi G\rho}{c^3}(\vec{\omega}\times\vec{r})_i$$

Both sides have the same factor $(\vec{\omega}\times\vec{r})_i$, so we can reduce this to a scalar equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df(r)}{dr}\right) + \frac{2}{r}\frac{df(r)}{dr} = \frac{16\pi G\rho}{c^3}$$

Okay, if we can find a solution for $f(r)$ that satisfies this, the our ansatz for h_{0i} indeed is valid. The "trick" is to multiply both sides by r^4 and use the product rule to write:

$$r^4\left(\frac{1}{r^2}\underbrace{\frac{d}{dr}\left(r^2\frac{df(r)}{dr}\right)}_{r^2\frac{d^2f(r)}{dr^2}+2r\frac{df(r)}{dr}} + \frac{2}{r}\frac{df(r)}{dr}\right) = \frac{16\pi G\rho}{c^3}r^4$$

$$\Rightarrow \quad r^4\frac{d^2f(r)}{dr^2} + 3r^3\frac{df(r)}{dr} = \frac{16\pi G\rho}{c^3}r^4$$

$$\Rightarrow \quad \frac{d}{dr}\left(r^4\frac{df(r)}{dr}\right) = \frac{16\pi G\rho}{c^3}r^4$$

At this point, we need to solve this twice: both in the region outside the central mass, where $\rho = 0$, and in the region inside the mass, where $\rho =$ constant (for a *uniform* sphere). This gives us two different solutions, $f_{\text{in}}(r)$ and $f_{\text{out}}(r)$, both of which we find by integrating once with respect to r, diving by r^4 and integrating another time:

$$\begin{cases} \dfrac{d}{dr}\left(r^4\dfrac{df_{\text{in}}(r)}{dr}\right) = -\dfrac{16\pi G\rho}{c^3}r^4 \\ \dfrac{d}{dr}\left(r^4\dfrac{df_{\text{out}}(r)}{dr}\right) = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} f_{\text{in}}(r) = \dfrac{8}{5}\dfrac{\pi G\rho}{c^3}r^2 - \dfrac{A}{3r^3} + B \\ f_{\text{out}}(r) = -\dfrac{C}{3r^3} + D \end{cases}$$

Here, A, B, C and D are integration constants. To fix these, we now need *boundary conditions*. We'd like $f_{\text{in}}(r)$ to not blow up to infinity at $r = 0$, which forces $A = 0$.

For $f_{\text{out}}(r)$, just like in the previous section, we should have $f_{\text{out}}(r) \to 0$ as $r \to \infty$, which forces the constant $D = 0$. These regularity conditions then give us:

$$\begin{cases} f_{\text{in}}(r) = \dfrac{8}{5}\dfrac{\pi G\rho}{c^3}r^2 + B \\ f_{\text{out}}(r) = -\dfrac{C}{3r^3} \end{cases}$$

Now, for the actual boundary conditions, we should have these two solutions match at the boundary of our source region, $r = R$. For continuity, we should also have the derivatives match at $r = R$. This gives us two equations that fix B and C, namely:

$$\begin{cases} f_{\text{in}}(R) = f_{\text{out}}(R) \\ \dfrac{df_{\text{in}}(r)}{dr}\bigg|_{r=R} = \dfrac{df_{\text{out}}(r)}{dr}\bigg|_{r=R} \end{cases} \Rightarrow \begin{cases} B = -\dfrac{8}{3}\dfrac{\pi G\rho}{c^3}R^2 \\ C = \dfrac{16}{5}\dfrac{\pi G\rho}{c^3}R^5 \end{cases}$$

We then have the general solutions for h_{0i} both inside and outside our central mass:

$$h_{0i} = \begin{cases} h_{0i}^{\text{in}} = f_{\text{in}}(r)(\vec{\omega} \times \vec{r})_i = -\dfrac{8}{3}\dfrac{\pi G\rho}{c^3}\left(R^2 - \dfrac{3}{5}r^2\right)(\vec{\omega} \times \vec{r})_i & , \ r \le R \\ h_{0i}^{\text{out}} = f_{\text{out}}(r)(\vec{\omega} \times \vec{r})_i = -\dfrac{16}{15}\dfrac{\pi G\rho R^5}{c^3}\dfrac{1}{r^3}(\vec{\omega} \times \vec{r})_i & , \ r > R \end{cases}$$

We can simplify these a bit by using the fact that the angular momentum for a uniform rotating sphere can be expressed as $\vec{J} = 8\pi\rho\omega R^5/15$ and its corresponding angular velocity as $\vec{\omega} = \omega\vec{J}/J = 15\vec{J}/\left(8\pi\rho R^5\right)$. After some straightforward algebra:

$$h_{0i} = \begin{cases} -\dfrac{5G}{c^3R^3}\left(1 - \dfrac{3r^2}{5R^2}\right)(\vec{J} \times \vec{r})_i & , \ r \le R \\ -\dfrac{2G}{c^3r^3}(\vec{J} \times \vec{r})_i & , \ r > R \end{cases}$$

We mainly care about the spacetime *outside* our rotating mass ($r > R$). The inside solution was only needed for fixing the integration constants through boundary conditions and to relate the solution to angular momentum (which we couldn't have done without also knowing the interior solution). For the outside region, we then have:

$$h_{0i} = -\frac{2G}{c^3 r^3}(\vec{J} \times \vec{r})_i$$

For our uniform, rotating spherical mass distribution, the Newtonian potential has the form $\phi = -GM/r$. Our general solution for the metric in a "block matrix" form is then:

$$g_{\mu\nu} = \begin{pmatrix} -1 + h_{00} & h_{0i} \\ h_{0i} & \delta_{ij} + h_{ij} \end{pmatrix} = \begin{pmatrix} -1 + \dfrac{2GM}{c^2 r} & -\dfrac{2G}{c^3 r^3}(\vec{J} \times \vec{r})_i \\ -\dfrac{2G}{c^3 r^3}(\vec{J} \times \vec{r})_i & \left(1 + \dfrac{2GM}{c^2 r}\right)\delta_{ij} \end{pmatrix}$$

This is called the **Lense-Thirring metric**, which describes the spacetime around a weakly rotating spherical mass. The diagonal components we can see are the same as for the standard weak-field metric around a static mass. All the new stuff is contained in the off-diagonal components, which depend on the angular momentum of the central mass.

These off-diagonal components represent a relativistic phenomenon called **frame dragging**. To interpret this effect, we can first write down the line element generally as:

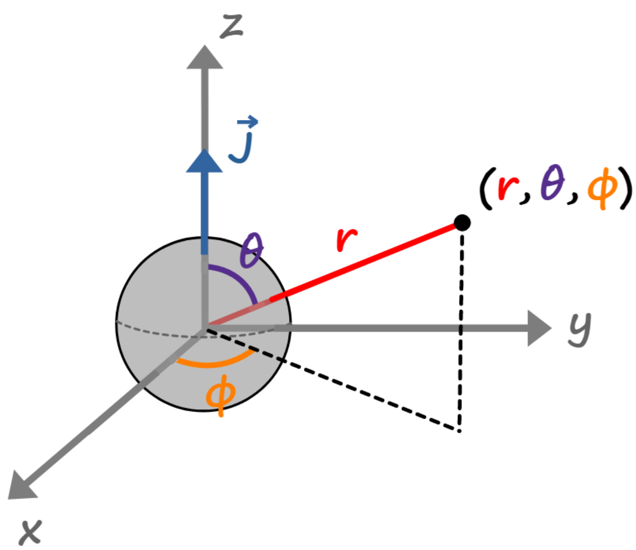
$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (-1 + h_{00})c^2 dt^2 + 2h_{0i} c dt dx^i + (\delta_{ij} + h_{ij}) dx^i dx^j$$

$$\Rightarrow \boxed{ds^2 = \left(-1 + \frac{2GM}{c^2 r}\right)c^2 dt^2 - \frac{4G}{c^3 r^3}(\vec{J} \times \vec{r})_i c dt dx^i + \left(1 + \frac{2GM}{c^2 r}\right)\delta_{ij} dx^i dx^j}$$

Next, let's express this explicitly in *spherical coordinates*, $x^\mu = \left(x^0, x^i\right) = (ct, r, \theta, \varphi)$. The dt^2-term will remain in exactly the same form. The $dx^i dx^j$-term will become, explicitly, $\delta_{ij} dx^i dx^j = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$ (this is just the usual spherical coordinate form).

What remains is to work out the $dtdx^i$-crossterm in spherical coordinates. To do that, we'll arrange our coordinate system so that the central mass is rotating around the z-axis. Its angular momentum vector is then $\vec{J} = J\vec{e}_z$. The components of $(\vec{J} \times \vec{r})_i$ become, using $\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_y$ as the position vector in Cartesian coordinates:

$$\vec{J} \times \vec{r} = J\vec{e}_z \times (x\vec{e}_x + y\vec{e}_y + z\vec{e}_z) = J\left(x \underbrace{\vec{e}_z \times \vec{e}_x}_{\vec{e}_y} + y \underbrace{\vec{e}_z \times \vec{e}_y}_{-\vec{e}_x} + z \underbrace{\vec{e}_z \times \vec{e}_z}_{0}\right) = J(x\vec{e}_y - y\vec{e}_x)$$



In Cartesian coordinates, the contraction $(\vec{J} \times \vec{r})_i dx^i$ would then be:

$$(\vec{J} \times \vec{r})_i dx^i = (\vec{J} \times \vec{r}) \cdot d\vec{x} = (\vec{J} \times \vec{r})_x dx + (\vec{J} \times \vec{r})_y dy + (\vec{J} \times \vec{r})_z dz = J(-ydx + xdy)$$

Next, we need to transform this to spherical coordinates. Because this only involves dx and dy (and no dz), we only need the Cartesian-to-spherical coordinate relations $x = r\sin\theta\cos\varphi$ and $y = r\sin\theta\sin\varphi$, which can be calculated using the chain rule:

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \varphi}d\varphi = \sin\theta\cos\varphi dr + r\cos\theta\cos\varphi d\theta - r\sin\theta\sin\varphi d\varphi$$
$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \varphi}d\varphi = \sin\theta\sin\varphi dr + r\cos\theta\sin\varphi d\theta + r\sin\theta\cos\varphi d\varphi$$

Using these now in our expression for $(\vec{J} \times \vec{r})_i dx^i$, we find a bunch of terms cancelling:

$$\begin{aligned}(\vec{J} \times \vec{r})_i dx^i &= J(-y dx + x dy) \\ &= J[-r \sin\theta \sin\varphi(\sin\theta\cos\varphi dr + r\cos\theta\cos\varphi d\theta - r\sin\theta\sin\varphi d\varphi) \\ &\quad + r\sin\theta\cos\varphi(\sin\theta\sin\varphi dr + r\cos\theta\sin\varphi d\theta + r\sin\theta\cos\varphi d\varphi)] \\ &= Jr^2 \sin^2\theta\left(\sin^2\varphi + \cos^2\varphi\right) d\varphi \\ &= Jr^2 \sin^2\theta d\varphi\end{aligned}$$

The Lense-Thirring metric, written as a line element in spherical coordinates, is then:

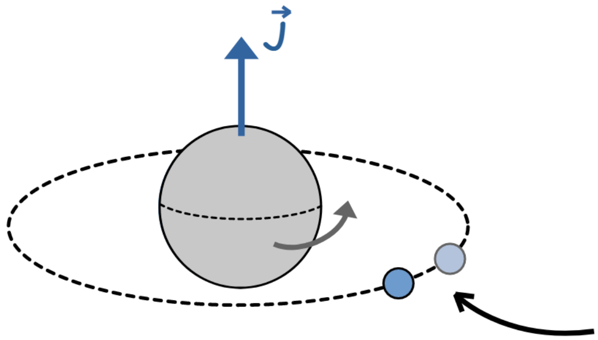
$$\boxed{ds^2 = \left(-1 + \frac{2GM}{c^2 r}\right) c^2 dt^2 - \frac{4GJ}{c^3 r}\sin^2\theta c dt d\varphi + \left(1 + \frac{2GM}{c^2 r}\right)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)}$$

If we wanted to, we could express the individual metric components in matrix form as well:

$$g_{\mu\nu} = \begin{pmatrix} -1 + \frac{2GM}{c^2 r} & 0 & 0 & -\frac{4GJ}{c^3 r}\sin^2\theta \\ 0 & 1 + \frac{2GM}{c^2 r} & 0 & 0 \\ 0 & 0 & \left(1 + \frac{2GM}{c^2 r}\right) r^2 & 0 \\ -\frac{4GJ}{c^3 r}\sin^2\theta & 0 & 0 & \left(1 + \frac{2GM}{c^2 r}\right) r^2 \sin^2\theta \end{pmatrix}$$

The interesting part here are the off-diagonal components $g_{t\varphi}$, which depend on the angular momentum J. These components represent **frame dragging**, in which, essentially, the spin of the central mass *drags* the entire spacetime around it. If we were to put a particle in orbit around it, it would get dragged along with the rotation of the central mass.

The hallmark of a frame dragging effect is usually some sort of mixed "time-angle", or $dtd\varphi$-component in the metric which we can see here (the same turns out to be true around a rotating black hole, described by the Kerr metric). The effect is strongest at the equatorial plane at $\theta = \pi/2$ and vanishes at the poles $\theta = 0$ and $\theta = \pi$.



Free particles are "dragged" along with the rotation of the central body

Frame Dragging & Lense-Thirring Precession

Next, we will analyze some interesting effects that appear in the Lense-Thirring spacetime, including frame dragging, spin-orbit coupling and precession effects. These effects appear in many other spacetimes as well, so understanding them in our much simpler, "almost-Newtonian" setting here will be very worthwhile.

With that said, our starting point is to assume we have a particle with mass m in the Lense-Thirring spacetime. Its coordinates, (r, θ, φ), are described by the equations of motion (under the assumptions that M, J and the particle's velocity are all small):

$$\begin{cases} \ddot{r} = -\dfrac{GM}{r^2} + r\dot{\theta}^2 + r\sin^2\theta\dot{\varphi}^2 + \dfrac{2GJ}{c^2r^2}\sin^2\theta\dot{\varphi} \\ \ddot{\theta} = -\dfrac{2}{r}\dot{r}\dot{\theta} + \sin\theta\cos\theta\dot{\varphi}^2 - \dfrac{4GJ}{c^2r^3}\sin\theta\cos\theta\dot{\varphi} \\ \dfrac{d}{dt}\left(mr^2\sin^2\theta\dot{\varphi} - \dfrac{2GJm}{c^2r}\sin^2\theta\right) = 0 \end{cases}$$

These can be derived from the geodesic equation, which will be the topic of Lesson 13.

Here, the dots denote time derivatives, so for example, $\dot{r} = dr/dt$ and $\ddot{r} = d^2r/dt^2$. The last equation is the equation of motion for the φ-coordinate, which actually describes the **conservation of angular momentum** of the particle. The quantity inside the parentheses is defined as the orbital angular momentum L, which we often write in terms of $\ell = L/m$, the orbital angular momentum per unit mass:

$$\ell = r^2 \sin^2\theta\dot{\varphi} - \frac{2GJ}{c^2 r}\sin^2\theta, \text{ with } \dot{\ell} = 0.$$

This allows us to write $\dot{\varphi}$ completely in terms angular momenta in the form:

$$\dot{\varphi} = \frac{\ell}{r^2 \sin^2\theta} + \frac{2GJ}{c^2 r^3}$$

We can see that the angular speed of the particle, $\dot{\varphi}$, depends on both the particle's own angular momentum (ℓ) but also the angular momentum of the central mass (J). So, somehow the central mass rotating also contributes to the actual rotation speed of the particle. This is an effect of frame dragging, which you can interpret as the central mass *dragging* spacetime along with is rotation and causing the particle's rotation to be either sped up or slowed down (depending on the relative signs of ℓ and J).

Another way to understand this effect is that the particle having zero angular momentum no longer means it is *stationary*. If we take a particle with $\ell = 0$, it will remain at that value because ℓ is conserved. However, *even though* the particle has zero orbital angular momentum, it will still have a non-zero angular speed $\dot{\varphi} \equiv \omega$:

$$\dot{\varphi} = \frac{\ell}{r^2 \sin^2\theta} + \frac{2GJ}{c^2 r^3} \quad \overset{\ell=0}{\Rightarrow} \quad \omega = \frac{2GJ}{c^2 r^3}$$

The particle can essentially have no angular momentum yet still be in rotational motion. In more technical terms, we say that *local inertial frames*, or observers in free-fall with zero angular momentum are forced to rotate *with* the central mass. To actually remain stationary ($\dot{\varphi} = 0$) would require acceleration and a non-inertial frame.

In "everyday" situations, the effects of frame dragging are really small. For example, at an altitude of $100\ \mathrm{km}$ from Earth's surface, we would find $\omega \sim 4 \times 10^{-14}\ s^{-1}$, which means that in one year, the particle would only be dragged about 0.00007 degrees.

In terms of the equations of motion, the conservation of angular momentum allows us to eliminate the φ-coordinate if we insert the above expression for $\dot{\varphi}$ into them. When doing so, we'll get a term $\propto J^2$, which is strictly beyond our weak-field linear approximation. So, to be consistent, we should discard this term, and get:

$$\begin{cases} \ddot{r} \approx -\dfrac{GM}{r^2} + r\dot{\theta}^2 + \dfrac{\ell^2}{r^3 \sin^2\theta} + \dfrac{6GJ\ell}{c^2 r^4} \\ \ddot{\theta} \approx -\dfrac{2}{r}\dot{r}\dot{\theta} + \dfrac{\ell^2}{r^4}\dfrac{\cos\theta}{\sin^3\theta} \end{cases}$$

The interesting new thing here is the term $\propto J\ell$ in the radial equation. This represents a relativistic effect called **spin-orbit coupling**, which produces a "force-like" effect pushing the particle either radially inward or outward depending on the *sign* of $J\ell$.

We can also take a look at some special cases of solutions to these equations. An interesting one is an **equatorial circular orbit**, which has $\theta = \pi/2$ and $r = a$ (constant), so $\ddot{\theta} = \dot{\theta} = \ddot{r} = \dot{r} = 0$. For such an orbit, we could calculate the corrections to the *orbital period* T due to frame dragging. For this, it's best to use the r- and θ-equations of motion and define $\dot{\varphi} = \omega = 2\pi/T$, so they take the form:

$$\begin{cases} \ddot{r} = -\dfrac{GM}{r^2} + r\dot{\theta}^2 + r\underbrace{\sin^2\theta}_{=1}\dot{\varphi}^2 + \dfrac{2GJ}{c^2 r^2}\underbrace{\sin^2\theta}_{=1}\dot{\varphi} \\ \ddot{\theta} = -\dfrac{2}{r}\dot{r}\dot{\theta} + \sin\theta\underbrace{\cos\theta}_{=0}\dot{\varphi}^2 - \dfrac{4GJ}{c^2 r^3}\sin\theta\underbrace{\cos\theta}_{=0}\dot{\varphi} \end{cases}$$

$$\Rightarrow \quad \begin{cases} 0 = -\dfrac{GM}{a^2} + a\left(\dfrac{2\pi}{T}\right)^2 + \dfrac{2GJ}{c^2 a^2}\dfrac{2\pi}{T} \\ 0 = 0 \end{cases}$$

The θ-equation here is trivially satisfied, so we have only the radial equation left. This gives us a quadratic equation for the orbital period T, which has two solutions:

$$T = 2\pi\sqrt{\frac{J^2}{M^2 c^4} + \frac{a^3}{GM}} \pm \frac{2\pi J}{Mc^2}$$

The $\propto J^2$-term is once again beyond our slow-rotation approximation, so we should drop it. This gives the orbital period - or, effectively, Kepler's third law - as:

$$T \approx 2\pi\left(\sqrt{\frac{a^3}{GM}} \pm \frac{J}{Mc^2}\right)$$

So, frame dragging also enters as a correction to the orbital period. The solution with $+$ here corresponds to a *retro*grade orbit (against the spin of the central mass, so the orbital period is slightly longer), and the one with $-$ to a *pro*grade orbit with a shorter period. In Newtonian gravity, there would be no such split between pro- and retrograde orbits, so this is a purely relativistic effect. This difference in orbital periods of pro- and retrograde orbits is sometimes called the **gravitomagnetic clock effect**.

Another interesting phenomenon can be best understood by looking at a **polar circular orbit**. Such an orbit passes above both the North and South poles, so we could calculate then the z-component of the orbital angular momentum (ℓ) at one of the poles, for example, at the North pole $\theta = 0$:

$$\ell(r, \theta = 0, \varphi) = r^2\dot{\varphi}\sin^2(0) - \frac{2GJ}{c^2 r}\sin^2(0) = 0$$

Because this is a conserved quantity, its value stays the same at all times, so $\ell = 0$ everywhere else along the polar orbit. A circular orbit also has $r = a$, and along with the result that $\ell = 0$ we just concluded, we will get for the φ-coordinate:

$$\dot{\varphi} = \frac{\ell}{r^2\sin^2\theta} + \frac{2GJ}{c^2r^3} = \frac{2GJ}{c^2a^3} \quad \Rightarrow \quad \varphi(t) = \Omega t + \varphi_0, \text{ where } \Omega \equiv \frac{2GJ}{c^2a^3}.$$

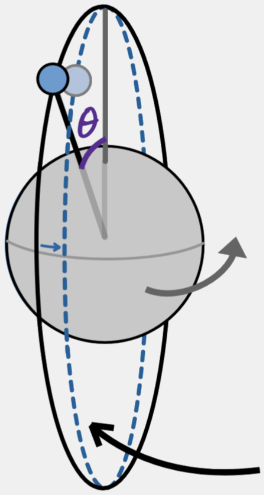
We'll interpret this soon. For the r- and θ-coordinates, we find with $r = a$ and $\ell = 0$:

$$\begin{cases} \ddot{r} = -\dfrac{GM}{r^2} + r\dot{\theta}^2 + \dfrac{\ell^2}{r^3\sin^2\theta} + \dfrac{6GJ\ell}{c^2r^4} \\ \ddot{\theta} = -\dfrac{2}{r}\dot{r}\dot{\theta} + \dfrac{\ell^2}{r^4}\dfrac{\cos\theta}{\sin^3\theta} \end{cases} \Rightarrow \begin{cases} 0 = -\dfrac{GM}{a^2} + a\dot{\theta}^2 \\ \ddot{\theta} = 0 \end{cases}$$

Notice that all the relativistic corrections are gone from these. Frame dragging around a central mass rotating around the z-axis only affects the φ-direction, so the r- and θ-motion remain unaffected (to first order) in a polar orbit. The solution for $\theta(t)$ from the above equations describes circular motion at a constant angular speed ($\theta(0) = 0$):

$$\theta(t) = \sqrt{\frac{GM}{a^3}}t$$

So, what do these solutions mean? We start off with a circular orbit at $r = a$ in the plane $\varphi(0) = \varphi_0$, where the particle orbits with angular speed $\omega = \sqrt{GM/a^3}$. However, the orbital plane itself (value of φ) undergoes *precession*, meaning it changes over time at a rate $\Omega = 2GJ/c^2a^3$. Frame dragging causes the entire orbital plane to shift slowly over time, which is called (nodal) **Lense-Thirring precession**.



Frame dragging causes the orbital plane to slowly **shift over time** → an effect called *Lense-Thirring precession*

In Newtonian gravity, nodal precession can happen only if the central body has some *oblateness* but not when it's a perfect sphere. Here, however, we've just found that it would happen *even* around a spherical body if the body is spinning. Lense-Thirring precession is a real and experimentally verifiable (though small) relativistic effect.

4. An Introduction To Gravitational Waves

The next topic we'll discuss are **gravitational waves**. We already got a hint of how "wave-like" solutions might arise from the linearized Einstein field equations in the Lorenz gauge, and the plan for this section is to continue up on that discussion.

In this section, we will mainly discuss gravitational waves as **vacuum solutions** to the field equations. I turns out that the Lorenz gauge does not fix such solutions completely, so there exists something called *residual gauge freedom*. After discussing this, we will develop the mathematics for describing the polarization modes of gravitational waves and also discuss some interesting applications like gravitational wave detectors.

In the next section, we'll go beyond the vacuum solutions discussed here, which involves solving the full, *sourced* Einstein field equations with $T_{\mu\nu} \neq 0$. Doing this will allow us to understand how a given matter configuration may or may not produce gravitational waves.

4.1. The Transverse-Traceless Gauge

Earlier, we derived the general form of the Einstein field equations in the Lorenz gauge and saw that they took the form of **wave equations** for the trace-reversed perturbation:

$$\partial^\alpha \partial_\alpha h^{\text{TR}}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \quad \Rightarrow \quad \nabla^2 h^{\text{TR}}_{\mu\nu} - \frac{1}{c^2}\frac{\partial^2 h^{\text{TR}}_{\mu\nu}}{\partial t^2} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

We can see that the energy-momentum tensor $T_{\mu\nu}$ acts as a source for whatever gravitational waves are described by these equations, just like current densities would for electromagnetic waves. The factor c^{-2} here represents the fact that gravitational waves travel at the *speed of light*. These waves are able to propagate in a *vacuum*, as the above equation reduces to a vacuum wave equation when no sources are present: $\partial^\alpha \partial_\alpha h^{\text{TR}}_{\mu\nu} = 0$.

Now, we don't know yet which components of $h^{\text{TR}}_{\mu\nu}$ actually contain these gravitational waves. To find that out, we need to discuss a feature called **residual gauge symmetry**.

The idea is that the Lorenz gauge, $\partial^\mu h^{\mathrm{TR}}_{\mu\nu} = 0$, does *not* actually completely fix our gauge yet. If we were to do an additional gauge transformation to the Lorenz gauge condition itself, $\overline{h}^{\mathrm{TR}}_{\mu\nu} = h^{\mathrm{TR}}_{\mu\nu} - \partial_\mu \xi^{\mathrm{R}}_\nu - \partial_\nu \xi^{\mathrm{R}}_\mu + \eta_{\mu\nu}\partial^\alpha \xi^{\mathrm{R}}_\alpha$, it would transform as:

$$
\begin{aligned}
&\partial^\mu \overline{h}^{\mathrm{TR}}_{\mu\nu} = 0 \\
&\Rightarrow \quad \partial^\mu \left(h^{\mathrm{TR}}_{\mu\nu} - \partial_\mu \xi^{\mathrm{R}}_\nu - \partial_\nu \xi^{\mathrm{R}}_\mu + \eta_{\mu\nu}\partial^\alpha \xi^{\mathrm{R}}_\alpha \right) = 0 \\
&\Rightarrow \quad \partial^\mu h^{\mathrm{TR}}_{\mu\nu} - \partial^\mu \partial_\mu \xi^{\mathrm{R}}_\nu - \partial_\nu \partial^\mu \xi^{\mathrm{R}}_\mu + \underbrace{\eta_{\mu\nu}\partial^\mu}_{=\partial_\nu} \underbrace{\partial^\alpha \xi^{\mathrm{R}}_\alpha}_{=\partial^\mu \xi^{\mathrm{R}}_\mu} = 0 \\
&\Rightarrow \quad \partial^\mu h^{\mathrm{TR}}_{\mu\nu} - \partial^\mu \partial_\mu \xi^{\mathrm{R}}_\nu - \partial_\nu \partial^\mu \xi^{\mathrm{R}}_\mu + \partial_\nu \partial^\mu \xi^{\mathrm{R}}_\mu = 0 \\
&\Rightarrow \quad \partial^\mu \partial_\mu \xi^{\mathrm{R}}_\nu = \partial^\mu h^{\mathrm{TR}}_{\mu\nu}
\end{aligned}
$$

The notation ξ^{R}_ν here stands for a 'residual' gauge parameter.

If we now pick this residual gauge parameter to satisfy $\partial^\mu \partial_\mu \xi^{\mathrm{R}}_\nu = 0$, the above equation would give us $\partial^\mu h^{\mathrm{TR}}_{\mu\nu} = 0$ - which means we would *still* be in the Lorenz gauge, even after an additional gauge transformation. So, the Lorenz gauge actually allows us to do an additional but more restricted **residual gauge transformation** - more restricted because the gauge parameter now has to also satisfy a wave equation of the form $\partial^\mu \partial_\mu \xi^{\mathrm{R}}_\nu = 0$.

This residual gauge symmetry highlights redundancy in the definition of $h_{\mu\nu}$. Originally, the Lorenz gauge would use up our *four* independent gauge conditions, which reduces the maximum number of independent components of $h_{\mu\nu}$ from 10 to 6. Residual gauge symmetry means that there is still gauge freedom left, in fact, for imposing *four more* conditions on the perturbation. So, it truly only contains 6 - 4 = **2 degrees of freedom**.

Sidenote: earlier, when we solved the Newtonian limit in the *stationary* case, there was *no* residual gauge symmetry left. This is because the condition $\partial^\mu \partial_\mu \xi^{\mathrm{R}}_\nu = 0$ reduces to $\nabla^2 \xi^{\mathrm{R}}_\nu = 0$ for the time-independent case, which has the unique solution $\xi^{\mathrm{R}}_\nu = 0$. So, residual gauge symmetry only appears for a time-dependent metric.

We can now exploit this to bring out the *radiative* components of gravitational waves. The appropriate residual gauge choice that does this is called the **transverse-traceless gauge**.

What does this gauge do? First, it transforms the trace-reversed metric $h_{\mu\nu}^{\mathrm{TR}}$ to a new metric, which we'll label as $h_{\mu\nu}^{\mathrm{TT}}$. The '*transverse*'-part of this gauge sets $h_{0\mu}^{\mathrm{TT}} = 0$ for all μ, which will pick out those components that are **transverse** (orthogonal) to the direction of wave propagation. The '*traceless*'-part sets $h^{\mathrm{TT}} = \eta^{\mu\nu} h_{\mu\nu}^{\mathrm{TT}} = \delta^{ij} h_{ij}^{\mathrm{TT}} = 0$ (the second equality comes from the transverse condition setting $h_{00}^{\mathrm{TT}} = 0$). The trace being zero can be thought of as a consequence of *vacuum* gravitational waves being **volume-preserving**, so they stretch spacetime along one direction, while squeezing it in another.

Both of these conditions (transverse and traceless) represent *physical* features of gravitational waves we expect to be there in any gauge. The TT-gauge just directly brings out these features, so it's a convenient gauge choice for describing gravitational waves.

The TT-gauge conditions

$$\begin{cases} h_{0\mu}^{\mathrm{TT}} = 0 \\ \delta^{ij} h_{ij}^{\mathrm{TT}} = 0 \end{cases}$$

In this gauge, the *trace-reversed* perturbation reduces to $h_{\mu\nu}^{\mathrm{TR}} = h_{\mu\nu}^{\mathrm{TT}} - h^{\mathrm{TT}}/2 = h_{\mu\nu}^{\mathrm{TT}}$. So, the trace-reversed and the original perturbation become the same thing in this gauge.

Now, the transverse-traceless gauge only physically makes sense when the *dominant* form of gravity is gravitational radiation. This is usually satisfied when we are really far away from any matter sources, in the so-called *far-field region*. For this reason, the TT-gauge is said to be a **radiation gauge**, and not a *global* gauge: it's not applicable everywhere, and it also requires a vacuum region (so, $T_{\mu\nu} = 0$) with no static gravitational field present.

Are there any other conditions we need? Well, the TT-gauge is a **special case of the Lorenz gauge**, so we of course also need the Lorenz gauge to hold: $\partial^\mu h_{\mu\nu}^{\mathrm{TT}} = 0$. The $\nu = 0$ -part of this is satisfied automatically when $h_{\mu 0}^{\mathrm{TT}} = 0$, and the $\nu = j$ -part leads to:

$$\partial^\mu h_{\mu j}^{\mathrm{TT}} = \partial^0 \underbrace{h_{0j}^{\mathrm{TT}}}_{\equiv 0} + \partial^i h_{ij}^{\mathrm{TT}} = 0 \quad \Rightarrow \quad \boxed{\partial^i h_{ij}^{\mathrm{TT}} = 0}$$

This is an additional "consistency condition" we need in order for both the Lorenz gauge and the TT-gauge to be satisfied.

Now, like before, we don't need to know the residual gauge parameter that puts us in the TT-gauge explicitly. Simply knowing one *exists* is enough to impose it, and this can be done for any *purely radiative* metric perturbation. In practice, we usually do it in two steps:

1. **Solve the Lorenz gauge (vacuum) wave equation**, $\partial^\alpha \partial_\alpha h_{\mu\nu}^{\mathrm{TR}} = 0$, for some initial trace-reversed metric. This might have the form $h_{\mu\nu}^{\mathrm{TR}} \sim \cos\left(k^\alpha x_\alpha\right) + \sin\left(k^\alpha x_\alpha\right)$, and the Lorenz condition, $\partial^\mu h_{\mu\nu}^{\mathrm{TR}} = 0$, gives you restrictions on its **amplitudes**.

2. **Impose the transverse-traceless gauge**, which now demands that:

$$\left.\begin{aligned} h_{0\mu}^{\mathrm{TT}} &= 0 \\ \partial^i h_{ij}^{\mathrm{TT}} &= 0 \end{aligned}\right\} \begin{array}{l} \textbf{The transverse conditions} \\ \text{(the second one comes from the Lorenz gauge)} \end{array}$$

$$\delta^{ij} h_{ij}^{\mathrm{TT}} = 0 \qquad \textbf{The traceless condition}$$

In practice, these conditions can all be recast as *algebraic* restrictions on the amplitudes of the solution from step #1.

4.2. Vacuum Gravitational Waves

In this section, we'll find our first *wave* solution to the vacuum field equations. After that, we'll impose the traceless-transverse gauge and see what kinds of additional restrictions the solution must satisfy. Let's begin by writing the wave equation in the explicit form:

$$\begin{aligned} &\partial^\alpha \partial_\alpha h_{\mu\nu}^{\mathrm{TR}} = 0 \\ &\Rightarrow \quad \partial^0 \partial_0 h_{\mu\nu}^{\mathrm{TR}} + \partial^i \partial_i h_{\mu\nu}^{\mathrm{TR}} = 0 \\ &\Rightarrow \quad \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) h_{\mu\nu}^{\mathrm{TR}} - \frac{1}{c^2} \partial_t h_{\mu\nu}^{\mathrm{TR}} = 0 \end{aligned}$$

Let's try a solution ansatz for this. We'll take $h_{\mu\nu}^{\mathrm{TR}} = \epsilon_{\mu\nu} \cos(\omega t - kz)$, which describes a plane wave moving along the z-axis with frequency $\omega = 2\pi f$ and wavenumber $k = 2\pi / \lambda$. It has an amplitude $\epsilon_{\mu\nu}$ that is a *second-rank* tensor. This tensor is called the **polarization tensor**, as it represents the "directions" in which the field $h_{\mu\nu}^{\mathrm{TR}}$ oscillates in.

Plugging this ansatz into the wave equation now gives us:

$$\Big(\underbrace{\partial_x^2}_{=0} + \underbrace{\partial_y^2}_{=0} + \partial_z^2\Big)\epsilon_{\mu\nu}\cos(\omega t - kz) - \frac{1}{c^2}\epsilon_{\mu\nu}\partial_t^2\cos(\omega t - kz) = 0$$
$$\Rightarrow \quad -k^2\epsilon_{\mu\nu}\cos(\omega t - kz) + \frac{\omega^2}{c^2}\epsilon_{\mu\nu}\cos(\omega t - kz) = 0$$
$$\Rightarrow \quad \omega = ck$$

The wave equation can therefore be satisfied as long as the frequency and wavenumber are related by $\omega = ck$, which is called a **dispersion relation**. Electromagnetic plane waves, if you're familiar with them, also have precisely this kind of dispersion relation in a vacuum.

Now, a more "relativistic" way to write our plane wave is to collect both the frequency and the wavenumber into a four-vector, which has the form $k^\mu = (\omega/c, 0, 0, k) = (k, 0, 0, k)$, where $\omega = ck$ was used. Then, the phase argument of the wave can be written in the form:

$$\omega t - kz = \frac{\omega}{c}ct - kz = -k^0 x_0 + k^i x_i = k^\alpha x_\alpha, \text{ where } x_\alpha = (-ct, 0, 0, z).$$

Reminder; when lowering indices, time components pick up a minus sign from the Minkowski metric, which is why $x_0 = -x^0 = -ct$. The spatial coordinates are just $x^i = x_i = (0, 0, z)$.

Our wave solution can then be expressed in this form:

$$\boxed{h_{\mu\nu}^{\mathrm{TR}} = \epsilon_{\mu\nu}\cos\left(k^\alpha x_\alpha\right)}$$

This holds for a plane wave travelling in any direction, so we could generally have $k^\mu = (k, k_x, k_y, k_z) = \left(k, k^i\right)$, with $k = \sqrt{k^i k_i}$. Another nice thing about this is that taking derivatives is now rather simple. Consider, for example, the ∂^β-derivative of this:

$$\partial^\beta h_{\mu\nu}^{\mathrm{TR}} = \epsilon_{\mu\nu}\partial^\beta\cos\left(k^\alpha x_\alpha\right) = -\epsilon_{\mu\nu}\sin\left(k^\alpha x_\alpha\right)\underbrace{\partial^\beta\left(k^\alpha x_\alpha\right)}_{k^\alpha\delta_\alpha^\beta = k^\beta} = -k^\beta\epsilon_{\mu\nu}\sin\left(k^\alpha x_\alpha\right)$$

So, any derivative just changes the cosine to a (minus) sine and brings a factor of the wave four-vector in front. The wave equation itself could therefore be written as:

$$\begin{aligned}
&\partial^\alpha \partial_\alpha h_{\mu\nu}^{\mathrm{TR}} = 0 \\
&\Rightarrow \quad -k^\alpha k_\alpha \epsilon_{\mu\nu} \cos\left(k^\alpha x_\alpha\right) = 0 \\
&\Rightarrow \quad \boxed{k^\alpha k_\alpha = 0}
\end{aligned}$$

This says the wave four-vector is **null**, meaning it has *zero length* in spacetime. If you write this out component-wise, it's equivalent to the dispersion relation $\omega = ck$. For this reason, we say anything moving at the speed of light travels along *null paths* through spacetime.

Okay, we've reduced the linearized Einstein field equations to just the simple algebraic relation $k^\alpha k_\alpha = 0$. What about the **Lorenz gauge**? We also need it to be satisfied:

$$\begin{aligned}
&\partial^\mu h_{\mu\nu}^{\mathrm{TR}} = 0 \\
&\Rightarrow \quad -k^\mu \epsilon_{\mu\nu} \sin\left(k^\alpha x_\alpha\right) = 0 \\
&\Rightarrow \quad \boxed{k^\mu \epsilon_{\mu\nu} = 0}
\end{aligned}$$

This is basically a "dot product $k \cdot \epsilon = 0$" between the wave vector and the polarization tensor, which says these two must be *orthogonal*. This demands that the polarization modes of a plane gravitational wave should be **transverse** to its travel direction - $\epsilon_{\mu\nu}$ is orthogonal to k^μ . This is a general feature of electromagnetic plane waves as well.

Now, let's explicitly write out the summation over μ and use the fact that $k^0 = \omega / c = k$ (from the dispersion relation). This gives, for both $\nu = 0$ and $\nu = i$ individually:

$$\begin{cases} k^\mu \epsilon_{\mu 0} = k^0 \epsilon_{00} + k^i \epsilon_{i0} = k \epsilon_{00} + k^i \epsilon_{i0} = 0 \\ k^\mu \epsilon_{\mu i} = k^0 \epsilon_{0i} + k^j \epsilon_{ji} = k \epsilon_{0i} + k^j \epsilon_{ij} = 0 \end{cases} \Rightarrow \begin{cases} \epsilon_{00} = -\dfrac{k^i}{k} \epsilon_{i0} \\ \\ \epsilon_{0i} = -\dfrac{k^j}{k} \epsilon_{ij} \end{cases}$$

Or, further inserting ϵ_{0i} into the expression for ϵ_{00} (note that here $\epsilon_{0i} = \epsilon_{i0}$):

$$\epsilon_{00} = \frac{k^i k^j}{k^2} \epsilon_{ij}$$

This tells us that the polarization is entirely determined by ϵ_{ij} - once we know ϵ_{ij} and the (spatial) wave vector k^i, we can determine both ϵ_{0i} and ϵ_{00} and consequently the full wave solution too. The polarization components $\epsilon_{0\mu}$ are therefore actually *redundant*.

So far, we've only imposed the Lorenz gauge. Let's next impose the transverse-traceless gauge, which means we set $h_{0\mu}^{\mathrm{TT}} = 0$ and $\delta^{ij} h_{ij}^{\mathrm{TT}} = 0$. For our wave solution, these become algebraic conditions for the **polarization tensor**, namely, $\epsilon_{0\mu}^{\mathrm{TT}} = 0$ and $\delta^{ij}\epsilon_{ij}^{\mathrm{TT}} = 0$. We also have condition $\partial^i h_{ij}^{\mathrm{TT}} = 0$, which turns into $k^i \epsilon_{ij}^{\mathrm{TT}} = 0$. By imposing all of these, we're left with only the *traceless* and *transverse spatial* components h_{ij}^{TT}:

The TT-gauge conditions

(for the polarization tensor)

$$\omega = ck \qquad \epsilon_{0\mu}^{\mathrm{TT}} = 0$$
$$k^i \epsilon_{ij}^{\mathrm{TT}} = 0 \qquad \delta^{ij}\epsilon_{ij}^{\mathrm{TT}} = 0$$

While we don't *need* to explicitly solve for the specific ξ_μ^{R} that puts us in the TT-gauge, it can be nice to see a concrete example of how this would work. This is worked out below.

Example: Finding a Residual Gauge Parameter In Practice

We should begin by working out the differential equations needed to solve for the residual gauge parameter in the first place. The transformation of $h_{\mu\nu}^{\mathrm{TR}}$ and its trace are given by (we derived these in the section discussing the Lorenz gauge):

$$\bar{h}_{\mu\nu}^{\mathrm{TR}} = h_{\mu\nu}^{\mathrm{TR}} - \partial_\mu \xi_\nu^{\mathrm{R}} - \partial_\nu \xi_\mu^{\mathrm{R}} + \eta_{\mu\nu}\partial^\alpha \xi_\alpha^{\mathrm{R}}$$
$$\bar{h}^{\mathrm{TR}} = h^{\mathrm{TR}} - 2\partial^\alpha \xi_\alpha^{\mathrm{R}}$$

Now if we want to impose the TT-gauge where $h_{\mu 0}^{\mathrm{TT}} = h^{\mathrm{TT}} = 0$, we require that the above transformed quantities, with $\nu = 0$ in the first expression, should equal zero:

$$\begin{cases} h^{\mathrm{TT}}_{\mu 0} = \bar{h}^{\mathrm{TR}}_{\mu 0} \equiv 0 \\ h^{\mathrm{TT}} = \bar{h}^{\mathrm{TR}} \equiv 0 \end{cases} \quad \Rightarrow \quad \boxed{\begin{cases} \partial_\mu \xi^{\mathrm{R}}_0 + \partial_0 \xi^{\mathrm{R}}_\mu - \eta_{\mu 0}\partial^\alpha \xi^{\mathrm{R}}_\alpha = h^{\mathrm{TR}}_{\mu 0} \\ \partial^\alpha \xi^{\mathrm{R}}_\alpha = \dfrac{1}{2} h^{\mathrm{TR}} \end{cases}}$$

These are again partial differential equations we need to solve for the residual gauge parameter ξ^{R}_μ. Because the TT-gauge comes from a residual gauge, we need the gauge parameter to also satisfy $\partial^\alpha \partial_\alpha \xi^{\mathrm{R}}_\mu = 0$. Since *any* solution ξ^{R}_μ satisfying all of these conditions will work, a convenient choice we could try is $\xi^{\mathrm{R}}_\mu = a_\mu \sin\left(k^\alpha x_\alpha\right)$, which solves the wave equation when $k^\alpha k_\alpha = 0$. Plugging this, and $h^{\mathrm{TR}}_{\mu\nu} = \epsilon_{\mu\nu} \cos\left(k_\alpha x^\alpha\right)$ from earlier, into our gauge condition equations, we find:

$$\begin{cases} \partial_\mu \xi^{\mathrm{R}}_0 + \partial_0 \xi^{\mathrm{R}}_\mu - \eta_{\mu 0}\partial^\alpha \xi^{\mathrm{R}}_\alpha = h^{\mathrm{TR}}_{\mu 0} \\ \partial^\alpha \xi^{\mathrm{R}}_\alpha = \dfrac{1}{2} h^{\mathrm{TR}} \end{cases}$$

$$\Rightarrow \begin{cases} k_\mu a_0 \cos\left(k^\alpha x_\alpha\right) + k_0 a_\mu \cos\left(k^\alpha x_\alpha\right) - \eta_{\mu 0} k^\alpha a_\alpha \cos\left(k^\alpha x_\alpha\right) = \epsilon_{\mu 0} \cos\left(k_\alpha x^\alpha\right) \\ k^\alpha a_\alpha \cos\left(k^\alpha x_\alpha\right) = \dfrac{1}{2} \eta^{\mu\nu} \epsilon_{\mu\nu} \cos\left(k_\alpha x^\alpha\right) \end{cases}$$

$$\Rightarrow \begin{cases} k_\mu a_0 - k a_\mu - \eta_{\mu 0} k^\alpha a_\alpha = \epsilon_{\mu 0} \\ k^\alpha a_\alpha = \dfrac{1}{2} \eta^{\mu\nu} \epsilon_{\mu\nu} \end{cases}$$

We can split the first equation into two separate equations for $\mu = 0$ and $\mu = i$, and also write out the summations over α (here, $\eta_{i0} = 0$ and $k^0 = -k_0 = k$):

$$\begin{cases} k_0 a_0 - k a_0 - \eta_{00}\left(k^0 a_0 + k^i a_i\right) = \epsilon_{00} \\ k_i a_0 - k a_i = \epsilon_{i0} \\ k^0 a_0 + k^i a_i = \dfrac{1}{2} \underbrace{\eta^{\mu\nu} \epsilon_{\mu\nu}}_{=\epsilon^\mu_\mu} \end{cases} \quad \Rightarrow \quad \begin{cases} -k a_0 + k^i a_i = \epsilon_{00} \\ k_i a_0 - k a_i = \epsilon_{i0} \\ k a_0 + k^i a_i = \dfrac{1}{2} \epsilon^\mu_\mu \end{cases}$$

Solving these now amounts to some straightforward algebra. For instance, we can solve the third equation for $k^i a_i$, insert it into the first equation and solve for a_0.

Then, inserting the result into the second equation and solving for a_i would give:

$$\begin{cases} a_0 = \dfrac{1}{4k}\left(\epsilon^{\mu}_{\mu} - 2\epsilon_{00}\right) \\ a_i = \dfrac{1}{k}(k_i a_0 - \epsilon_{i0}) \end{cases}$$

These now give us an explicit solution for the residual gauge parameter! Choosing these amplitudes for $\xi^{\mathrm{R}}_{\mu} = a_{\mu}\sin\left(k^{\alpha}x_{\alpha}\right)$, with $a_{\mu} = \begin{pmatrix} a_0 & a_i \end{pmatrix}$, is guaranteed to put us in the TT-gauge. We can check that explicitly, just to be sure:

$$\begin{aligned} h^{\mathrm{TT}}_{00} &= h^{\mathrm{TR}}_{00} - \partial_0\xi^{\mathrm{R}}_0 - \partial_0\xi^{\mathrm{R}}_0 + \eta_{00}\partial^{\alpha}\xi^{\mathrm{R}}_{\alpha} \\ &= \epsilon_{00}\cos\left(k^{\alpha}x_{\alpha}\right) + ka_0\cos\left(k^{\alpha}x_{\alpha}\right) - k^i a_i\cos\left(k^{\alpha}x_{\alpha}\right) \\ &= \epsilon_{00}\cos\left(k^{\alpha}x_{\alpha}\right) + \underbrace{\frac{k^i}{k}\epsilon_{i0}}_{-\epsilon_{00}}\cos\left(k^{\alpha}x_{\alpha}\right) = 0 \\ h^{\mathrm{TT}}_{i0} &= h^{\mathrm{TR}}_{i0} - \partial_i\xi^{\mathrm{R}}_0 - \partial_0\xi^{\mathrm{R}}_i + \eta_{i0}\partial^{\alpha}\xi^{\mathrm{R}}_{\alpha} \\ &= \epsilon_{i0}\cos\left(k^{\alpha}x_{\alpha}\right) - k_i a_0\cos\left(k^{\alpha}x_{\alpha}\right) + ka_i\cos\left(k^{\alpha}x_{\alpha}\right) \\ &= \epsilon_{i0}\cos\left(k^{\alpha}x_{\alpha}\right) - k_i a_0\cos\left(k^{\alpha}x_{\alpha}\right) + k_i a_0\cos\left(k^{\alpha}x_{\alpha}\right) - \epsilon_{i0}\cos\left(k^{\alpha}x_{\alpha}\right) = 0 \end{aligned}$$

$$\begin{aligned} h^{\mathrm{TT}} &= h^{\mathrm{TR}} - 2\partial^{\alpha}\xi^{\mathrm{R}}_{\alpha} \\ &= \epsilon^{\mu}_{\mu}\cos\left(k^{\alpha}x_{\alpha}\right) - 2ka_0\cos\left(k^{\alpha}x_{\alpha}\right) - 2k^i a_i\cos\left(k^{\alpha}x_{\alpha}\right) \\ &= 2\epsilon_{00}\cos\left(k^{\alpha}x_{\alpha}\right) + 2\underbrace{\frac{k^i}{k}\epsilon_{i0}}_{-\epsilon_{00}}\cos\left(k^{\alpha}x_{\alpha}\right) = 0 \end{aligned}$$

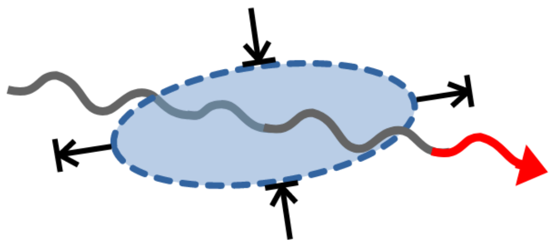
4.3. Gravitational Wave Polarization Modes

Let's now analyze the general relations we've found for gravitational plane waves. We can pick a wave travelling **in the z-direction**, so the wave vector has the form $k^i = (0, 0, k)$ (and $k^\mu = (k, 0, 0, k)$ from the dispersion relation). The choice of propagation direction is arbitrary, but specifying one explicitly just makes things much more concrete. With this choice of wave vector, the transverse-traceless conditions take the following form:

$$\begin{cases} \epsilon_{0\mu}^{\mathrm{TT}} = 0 \\ k^i \epsilon_{ij}^{\mathrm{TT}} = 0 \quad \Rightarrow \quad \epsilon_{zj}^{\mathrm{TT}} = 0 \\ \delta^{ij} \epsilon_{ij}^{\mathrm{TT}} = 0 \quad \Rightarrow \quad \epsilon_{xx}^{\mathrm{TT}} + \epsilon_{yy}^{\mathrm{TT}} + \epsilon_{zz}^{\mathrm{TT}} = 0 \end{cases}$$

The first two sets of conditions imply that any time- or z-components of the polarization tensor are zero, so the possible polarization "directions" are **transverse** to the z-direction. So, the perturbations the gravitational wave causes as it propagates occur in the xy-plane.

What about the traceless condition? With the transverse condition implying $\epsilon_{zz}^{\mathrm{TT}} = 0$, the tracelessness gives us $\epsilon_{yy}^{\mathrm{TT}} = -\epsilon_{xx}^{\mathrm{TT}}$. If the wave had, say, polarization in the x-direction (so, its amplitude oscillates in the x-direction), it must also have an equal but *opposite* polarization component in the y-direction. Essentially, a stretching of spacetime in one direction has to be accompanied by an equal squeezing in the "perpendicular" direction.



A **gravitational wave** **stretches** and simultaneously **squeezes** spacetime in the plane transverse to its **propagation direction**

We will denote here $\epsilon_{xx}^{\mathrm{TT}} = \epsilon_+$ and $\epsilon_{xy}^{\mathrm{TT}} = \epsilon_\times$ for reasons that will become more clear soon. The full polarization tensor, written as a 2×2 matrix, then takes the form:

$$\epsilon_{\mu\nu}^{\mathrm{TT}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_+ & \epsilon_\times & 0 \\ 0 & \epsilon_\times & -\epsilon_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We can see that there are only **two independent components**, which are encapsulated in the two polarization modes, ϵ_+ and $\epsilon_\times$. The metric having only two physical degrees of freedom is a result of the residual gauge symmetry we discussed earlier.

In electrodynamics, we find something similar in that residual gauge symmetry forbids a "longitudinal" polarization state for electromagnetic plane waves. In quantum electrodynamics, this means that photons only have two possible spin states (+1 or -1, but not 0!). In theories of **quantum gravity**, residual gauge symmetry similarly would prevent the graviton from having any other spin than +2 or -2.

With the polarization tensor for these gravitational waves solved and the perturbation given as $h_{\mu\nu}^{\mathrm{TT}} = \epsilon_{\mu\nu}^{\mathrm{TT}} \cos(\omega t - kz)$, we can then write down the full spacetime metric as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{\mathrm{TT}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 + \epsilon_+ \cos(\omega t - kz) & \epsilon_\times \cos(\omega t - kz) & 0 \\ 0 & \epsilon_\times \cos(\omega t - kz) & 1 - \epsilon_+ \cos(\omega t - kz) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let's now try to interpret what these two polarization modes actually describe. Why do we label them with "+" and " $\times$ "? Well, these labels come from simply how the deformation patterns of the two independent polarization modes "look" like - a *plus* or a *cross*. The diagonal cosine-terms here represent a **plus-polarized** wave and the off-diagonal ones a **cross-polarized** wave, hence the notation ϵ_+ and $\epsilon_\times$ for their amplitudes. The metric we have here represents a wave that is a general combination of both polarization modes.

To better understand these, we can write down the metric perturbations for both individual polarizations, so one gravitational wave that is purely *plus*-polarized and one wave that is purely *cross*-polarized. We get these simply by setting $\epsilon_+ = 0$ or $\epsilon_\times = 0$:

Plus-polarized gravitational wave ($\epsilon_\times = 0$, $\epsilon_+ \neq 0$):

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_+ \cos(\omega t - kz) & 0 & 0 \\ 0 & 0 & -\epsilon_+ \cos(\omega t - kz) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cross-polarized gravitational wave ($\epsilon_+ = 0$, $\epsilon_\times \neq 0$):

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_\times \cos(\omega t - kz) & 0 \\ 0 & \epsilon_\times \cos(\omega t - kz) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

These define the two transverse **linear polarization modes** of a gravitational wave. We can compare them to a linearly polarized electromagnetic wave along the z-direction. If the wave is linearly polarized in say the x-direction, its electric field is $\vec{E} \propto \vec{e}_x$. A linearly polarized gravitational wave does not have just a single polarization "axis", rather its linear polarization involves *tensor products* of basis directions, for example, for a *plus*-mode:

$$\vec{E} = E_0 \cos(\omega t - kz)\vec{e}_x \quad \Leftrightarrow \quad h = \epsilon_+ \cos(\omega t - kz)\vec{e}_x \otimes \vec{e}_x - \epsilon_+ \cos(\omega t - kz)\vec{e}_y \otimes \vec{e}_y$$

The difference to electromagnetic waves is simply that gravitational waves are not vector fields but **tensor fields**, so they don't have a simple "polarization axis" along any one direction. Instead, a gravitational wave requires two axes to oscillate along simultaneously.

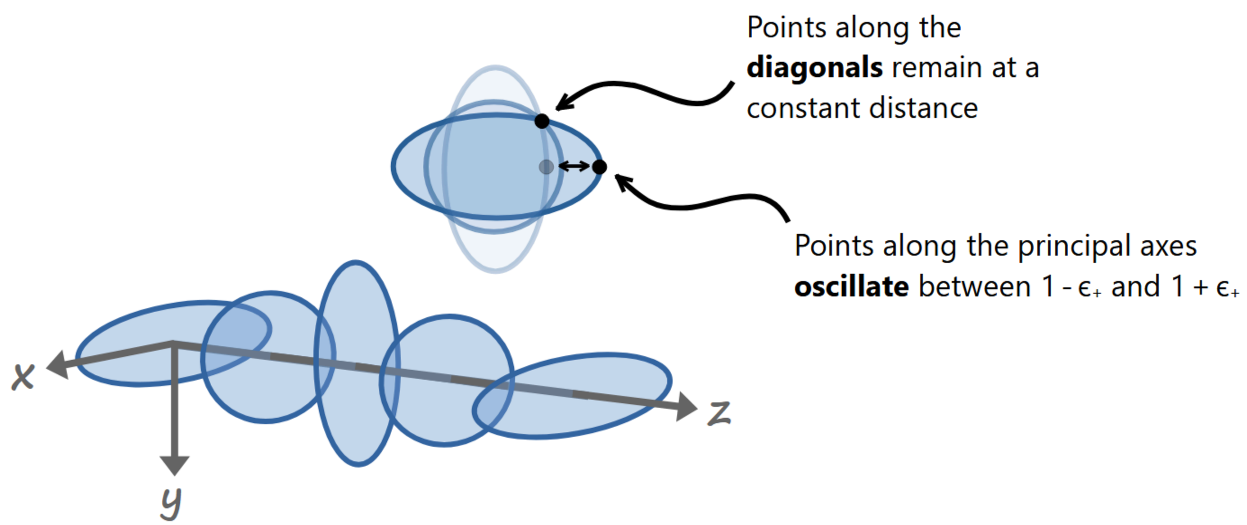
To understand what these polarization modes do, we can look at the line element, ds^2, which describes the coordinate-invariant distance between two nearby spacetime points.

Let's begin with a purely **plus-polarized** gravitational wave, which has the line element:

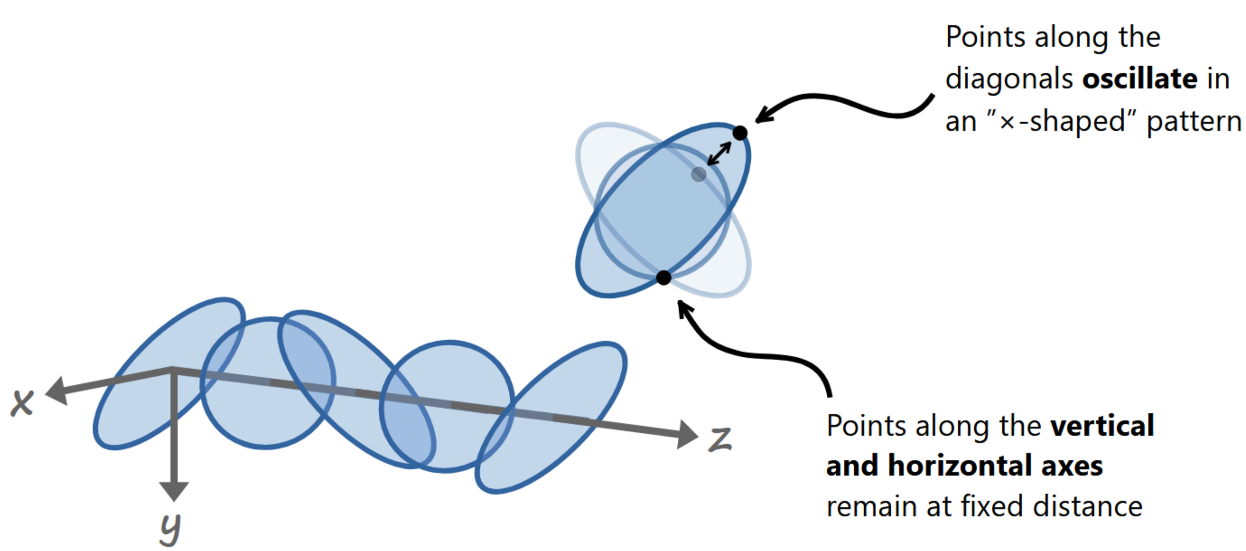
$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu = -c^2dt^2 + (1 + h_{11})dx^2 + (1 + h_{22})dy^2 + dz^2$$

$$\Rightarrow \boxed{ds^2 = -c^2dt^2 + [1 + \epsilon_+ \cos(\omega t - kz)]dx^2 + [1 - \epsilon_+ \cos(\omega t - kz)]dy^2 + dz^2}$$

Initially, at $z = t = 0$, this gives $(1 + \epsilon_+)dx^2$ and $(1 - \epsilon_+)dy^2$ at the same instant. These terms represent how lengths along the x- and y-directions are scaled, so the x-direction gets **stretched** by an amount ϵ_+ and the y-direction **squeezed** by the same amount. As time (and z) passes, this "squeeze-stretch" pattern will oscillate in a *plus-shaped* pattern:



The **cross-polarized** wave behaves in the same way but with its pattern rotated by $45°$:

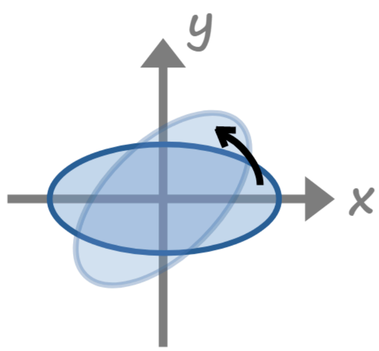


These depend on the orientation of our coordinate axes, so what looks like plus-polarization in one coordinate system might look like cross in another. What matters is that there are two *orthogonal* polarization modes, but not really which is defined as which.

We can also have any kind of combination of these two linear polarizations where both $\epsilon_+ \neq 0$ and $\epsilon_\times \neq 0$. One such special case is called a **circularly polarized gravitational wave**, which contains an equal amount of both linear polarizations, so $\epsilon_+ = \epsilon_\times \equiv \epsilon$:

$$\begin{aligned} h_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon\cos(\omega t - kz) & \epsilon\cos(\omega t - kz) & 0 \\ 0 & \epsilon\cos(\omega t - kz) & -\epsilon\cos(\omega t - kz) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon\cos(\omega t - kz) & 0 & 0 \\ 0 & 0 & -\epsilon\cos(\omega t - kz) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon\cos(\omega t - kz) & 0 \\ 0 & \epsilon\cos(\omega t - kz) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

This can be interpreted as a *rotating ellipse*, so there is no "up-down" oscillation pattern:



The most general type of plane gravitational wave is an **elliptically polarized** one, which is like a rotating ellipse but with some amount of "up-down" oscillation as well. The metric perturbation for this would be the same as above but with $\epsilon_+ \neq \epsilon_\times$.

So far, we've analyzed gravitational waves by picking the direction of propagation along the z-axis, however, everything discussed here would work the same for any other propagation direction. Later, we'll build a more general approach to describe this stuff, but before that, we can now take a look at an application: gravitational wave *detectors*.

4.4. Application: Gravitational Wave Detectors

Gravitational waves have been known about for a long time, but when they were first discovered, there was doubt if they could ever be detected due to their amplitudes (our $\epsilon_{+/\times}$) being so small. Indeed, it took almost a century for the first direct detection of a gravitational wave (in 2015), with many more having been detected since then.

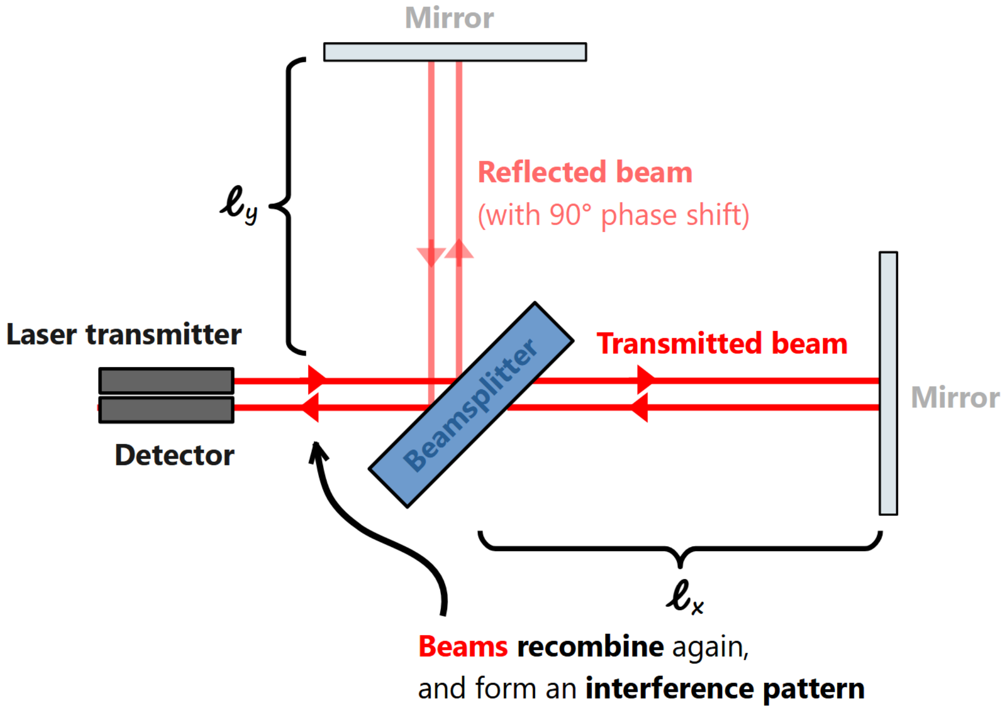
An interesting question to address is, how are these gravitational waves actually measured? We know that gravitational waves cause length changes, so at first glance, the answer would appear to be just "measure the change in length caused by such a wave".

But there is a problem: you cannot just go in with a ruler trying to measure such length changes because a gravitational wave passing by would *also* deform your ruler. There is no direct way to measure a length change if every local length changes simultaneously.

Currently existing detectors, such as LIGO, are instead based on **laser interferometry**. These measure the length change caused by a gravitational wave from an *interference pattern* of two laser beams, which *can* be measured unambiguously (either you have an interference pattern or you don't!). All kinds of information about the gravitational wave itself can then be extracted from this interference pattern.

Next, I'd like to walk you through how this process happens through a simplified example. We'll first look at the theory behind these laser interferometers and then how they relate to gravitational wave polarization patterns and the detection of these waves.

A laser interferometer for detecting gravitational waves should consist of two **perpendicular arms** that the laser beams travel along. This is because the oscillations a gravitational wave causes are maximal along the two perpendicular directions its polarization consists of. Below is roughly what one of these interferometers could look like.



Let's go through how this works. First, a laser beam is sent out to a **beamsplitter** where it gets splits into a reflected beam and a transmitted beam. A beamsplitter can be something as "simple" as a block consisting of many dielectric layers with suitably chosen permittivities. In our setup, we want the beamsplitter to be "50/50", meaning the incoming beam is split in *half* (equal power goes into both the reflected and transmitted beams).

The beamsplitter is usually engineered so that the *reflected* beam also picks up a **phase shift of 90°** compared to the transmitted beam. After the beamsplitter, we then have two equal amplitude beams along both arms, with a relative phase difference of 90° (i.e. $\pi/2$).

As the laser beam now travels along a given arm, it picks up a phase that depends on the length of the arm, say ℓ_x, and the wavenumber $k = \omega/c$ of the laser (which is an electromagnetic wave). Each beam bounces off the mirror and travels back the same path, so the total phases accrued along the arms are $2k\ell_x$ and $2k\ell_y$. Both beams then go through the beamsplitter again to combine at the detector side. The beam along the y-arm picks up another 90° phase shift here, so it is now out of phase by π compared to the other beam. The total **phase difference** as the beams combine at the detector is then:

$$\Delta\phi = 2k\ell_x - (2k\ell_y + \pi) = 2k(\ell_x - \ell_y) - \pi$$

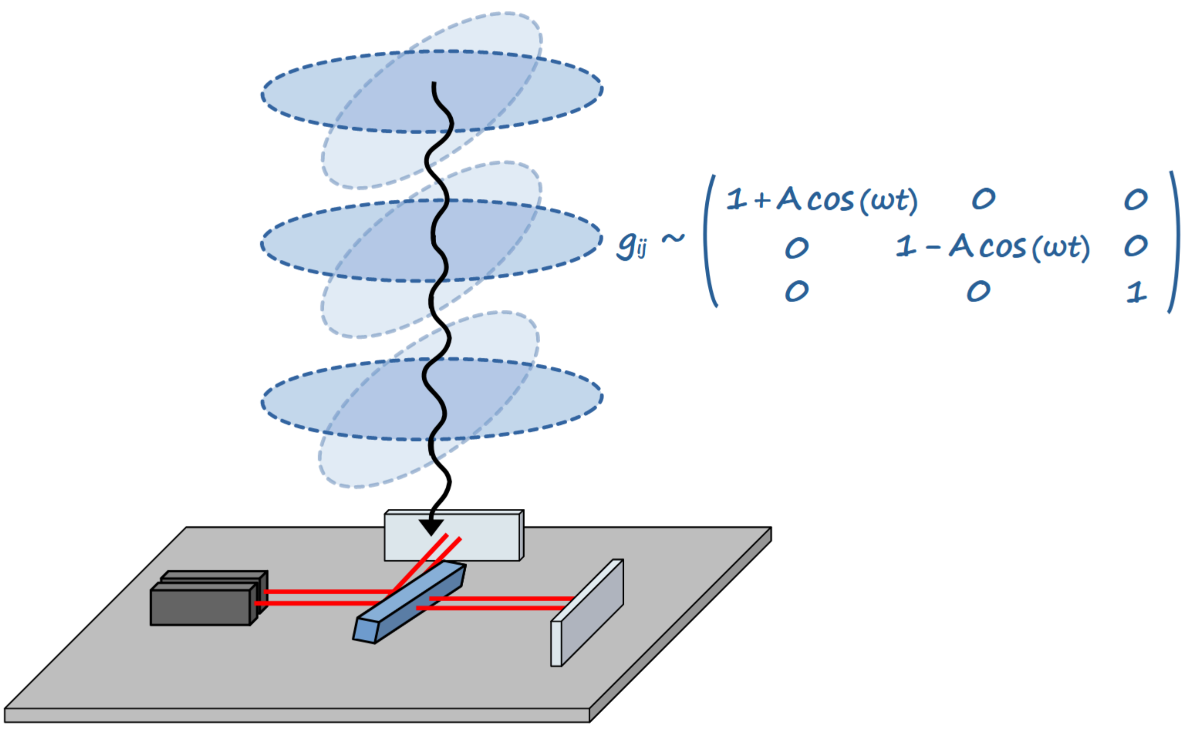
The detector then measures the total intensity of the combined beam, which is what we call an **interference pattern**. Depending on the phases of the two beams, they can interfere contructively or destructively, and the detector is only able to measure this total, interfered beam. The (normalized) interference pattern in this setup can be derived to be:

$$I = \cos^2\frac{\Delta\phi}{2} = \cos^2\left(\frac{2k(\ell_x - \ell_y) - \pi}{2}\right) = \sin^2[k(\ell_x - \ell_y)]$$

The purpose of the $90°$ phase shifts is to turn the cosine here into sine. The total intensity of the return laser now depends on the *length difference* of the arms. When there is *no* gravitational wave, we ideally don't want to detect anything. This is achieved if we choose the arms to have **equal length**, $\ell_x = \ell_y \equiv \ell_0$. When this is true, the resulting measured intensity will be zero, so the beams interfere *destructively* and cancel out at the detector.

What happens then when a gravitational wave passes by? By its polarization pattern, it will periodically stretch one of the arms and squeeze the other. This causes a non-zero length difference, $\ell_x - \ell_y \neq 0$, and therefore also a non-zero measured intensity at the detector.

Let's now consider a *plus-polarized* wave arriving along the z-axis, with the detector setup in the xy-plane at $z = 0$. We can take its plus-pattern to align with the detector arms.



The amplitude of the plus-polarized gravitational wave is denoted as the arbitrary constant A here.

The two relevant components of the metric at the detector plane ($z = 0$) are given by:

$$\begin{aligned} g_{11} &= 1 + A\cos(\omega t - kz) \overset{z=0}{=} 1 + A\cos\omega t \\ g_{22} &= 1 - A\cos(\omega t - kz) = 1 - A\cos\omega t \end{aligned}$$

We can calculate the length change caused by this wave on the two arms from the line element. At some particular time instance, the lengths of the x- and y-arms are given by integrating the line elements along these directions:

$$\begin{aligned} \ell_x &= \int_0^{\ell_0} ds_x = \int_0^{\ell_0} \sqrt{g_{11}}dx = \int_0^{\ell_0} \sqrt{1 + A\cos\omega t}dx \\ \ell_y &= \int_0^{\ell_0} ds_y = \int_0^{\ell_0} \sqrt{g_{22}}dy = \int_0^{\ell_0} \sqrt{1 - A\cos\omega t}dy \end{aligned}$$

We'll assume that the amplitude A is approximately constant across each arm (which is a good approximation as the wavelengths of measurable gravitational waves are on the order of $\sim 100\ \mathrm{km}$ or longer, whereas the arms of e.g. LIGO are "only" about $4\ \mathrm{km}$ long).

We also assume that A is small, so the square roots can be Taylor-expanded to first order:

$$\begin{aligned} \ell_x &= \int_0^{\ell_0} \sqrt{1 + A\cos\omega t}dx \approx \int_0^{\ell_0} \left(1 + \frac{A}{2}\cos\omega t\right)dx = \ell_0 + \frac{\ell_0 A}{2}\cos\omega t \\ \ell_y &= \int_0^{\ell_0} \sqrt{1 - A\cos\omega t}dy \approx \int_0^{\ell_0} \left(1 - \frac{A}{2}\cos\omega t\right)dx = \ell_0 - \frac{\ell_0 A}{2}\cos\omega t \end{aligned}$$

Because of this minus sign in ℓ_y, these lengths are notably *not* equal anymore due to the gravitational wave passing by. The measured intensity profile at the detector is then:

$$\begin{aligned} I &= \sin^2[k(\ell_x - \ell_y)] = \sin^2\left[k\left(\ell_0 + \frac{\ell_0 A}{2}\cos\omega t - \left(\ell_0 - \frac{\ell_0 A}{2}\cos\omega t\right)\right)\right] \\ &= \sin^2(k\ell_0 A\cos\omega t) \end{aligned}$$

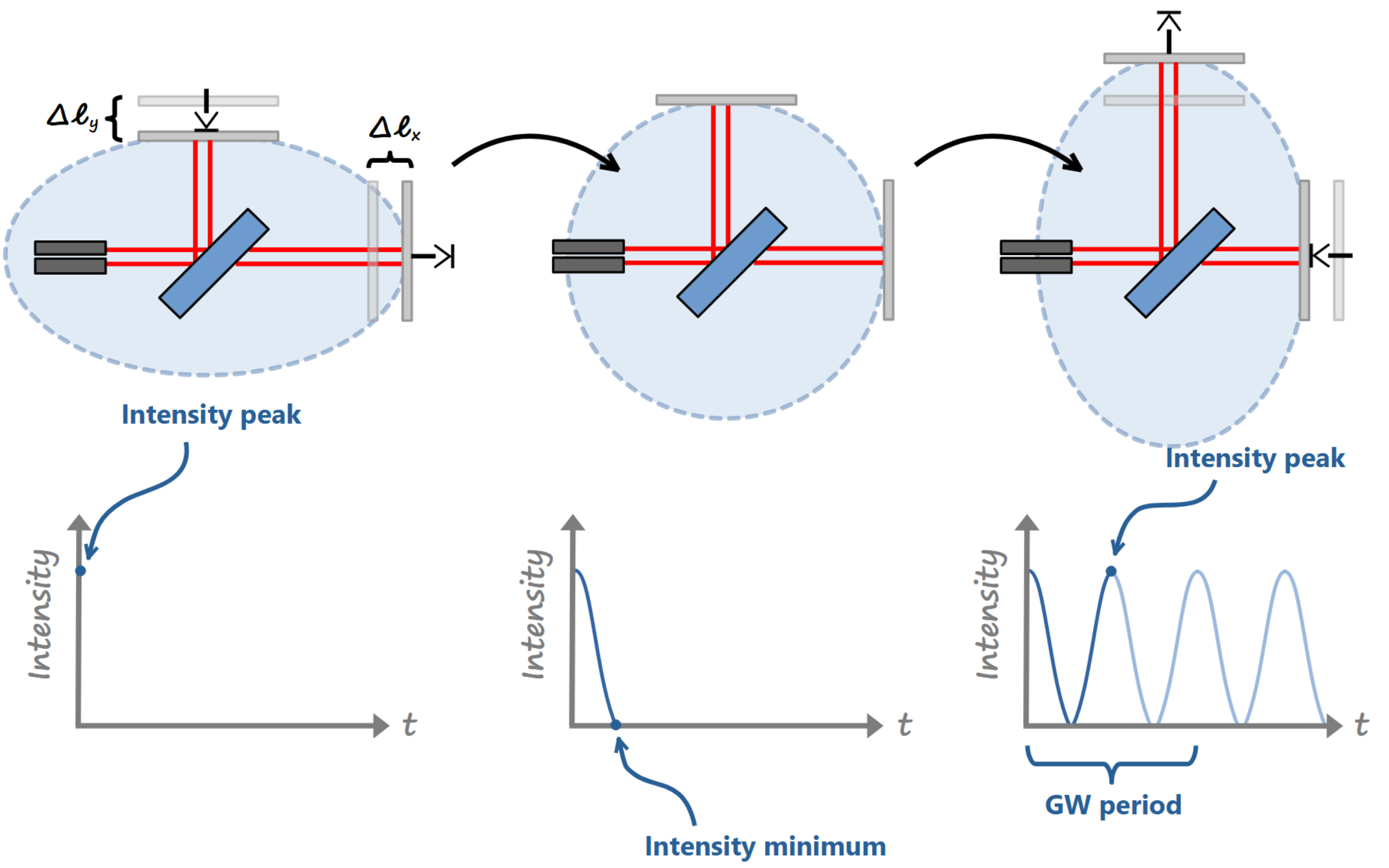
In any currently existing interferometer setup, the gravitational wave amplitude A is many orders of magnitude *smaller* than $k\ell_0 = 2\pi\ell_0 / \lambda$ (with λ the wavelength of the laser).

We can therefore use the small-angle approximation $\sin^2 x \approx x^2$ to get, finally:

$$\boxed{I(t) \approx (k\ell_0 A)^2 \cos^2 \omega t}$$

What does this tell us? Well, the intensity profile at the detector is now **time-dependent**. This is the main idea behind any gravitational wave detector: a gravitational wave passing by causes a detectable, time-varying interference pattern. The amplitude of these oscillations depends on the properties of the interferometer setup (laser wavenumber k and "equilibrium" length of the arms ℓ_0) and on the amplitude of the gravitational wave.

Also, the intensity profile, $I(t) \propto \cos^2 \omega t$, has *twice* the frequency of the gravitational wave (because $\cos^2 \omega t$ varies two times "faster" than $\cos \omega t$), so *half* the period. This makes sense because the detector itself is blind to which arm is longer and which one shorter at any given time - all it can see is the *difference* in arm length. This means then that, per gravitational wave period, there are two peaks in the intensity profile, one corresponding to the x-arm maximally stretched and one to the y-arm maximally stretched, respectively.



Real gravitational wave detectors like LIGO and the planned space detector LISA, are generally more complicated than our "toy" example here. For one, they have more complicated intensity readout methods. Because the amplitudes of gravitational waves passing by Earth are *tiny* (on the order of $\sim 10^{-21}\ \mathrm{m}$), many signal processing schemes are needed to distinguish actual gravitational waves from noise. For example, there are methods to make the intensity readout $\propto A\cos\omega t$ (and not its square).

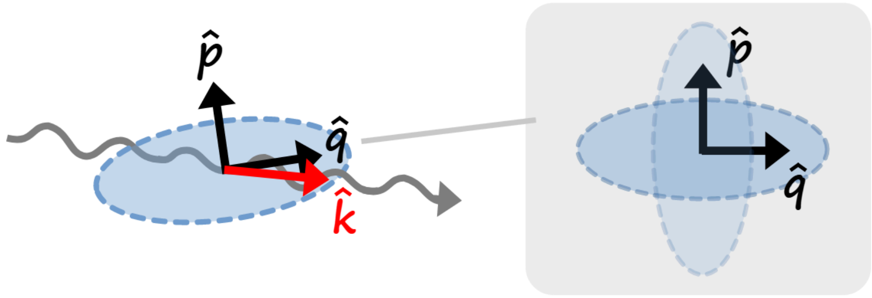
Another clever techniques is the use of **resonator cavities**. These are structures that allow the laser beam to bounce many times over along both arms before arriving at the detector. This has the effect of making the *effective* phase difference $\Delta\phi$ much larger.

4.5. Wave Frames & Polarization Basis Vectors

In this section, we will develop a more general framework for describing gravitational waves and their polarizations. This framework will turn out very useful when we get to computing gravitational waves from sources towards the end of this lesson.

Our goal, ultimately, is to extract the **transverse-traceless** (the "physical") part and the two polarization modes of a gravitational wave from a generic metric perturbation h_{ij}. This only requires knowing the direction of wave propagation (the wave vector components k^i), since we already know that the polarizations are transverse to it.

Our approach relies on constructing a **wave frame**, which is a basis $\{\hat{q}, \hat{p}, \hat{k}\}$ consisting of the wave propagation direction $\hat{k}$ (a unit vector) and two orthogonal transverse basis vectors $\hat{q}$ and $\hat{p}$. The unit vectors $\hat{q}$ and $\hat{p}$ are sometimes called the **transverse polarization basis**. The metric perturbation we know is transverse, so it has no components along the $\hat{k}$-axis. The here is to pick a basis "adapted" to the wave itself:



So, one of our coordinate axes is fixed by the wave propagation direction - but how do we choose $\hat{q}$ and $\hat{p}$? Well, their components q^i and p^i should satisfy the following properties:

- Normalization: $q_i q^i = p_i p^i = 1$
- Orthogonality: $q_i p^i = 0$
- Transversality: $q_i k^i = p_i k^i = 0$

As long as these are satisfied, we are essentially free to choose $\hat{q}$ and $\hat{p}$ however we see fit. For example, for a wave travelling in the z-direction, $\hat{k} = \vec{e}_z$, a convenient polarization basis choice could be just $\hat{q} = \vec{e}_x$ and $\hat{p} = \vec{e}_y$ (this is what we had in the previous section).

Now, we know that gravitational waves are always contained in the *transverse-traceless* part, h_{ij}^{TT}, of the spatial components h_{ij}. We could split h_{ij} as $h_{ij} = h_{ij}^{\mathrm{TT}} + (\text{other stuff})_{ij}$, so into the transverse-traceless part (which is what we want) and "$(\text{stuff})_{ij}$" containing its trace, longitudinal components along the wave propagation direction and other things.

We further know that the transverse-traceless part should consist of two independent components: the **plus- and cross-polarization modes.** In fact, the whole purpose of constructing our $\{\hat{q}, \hat{p}\}$-basis is that, in this basis, the TT-part would take the form:

$$h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The question now is, how do we extract the h_{ij}^{TT}-part from a generic set of spatial components h_{ij}? We can begin by writing the above matrix representation in the form of tensor products between $\hat{q}$ and $\hat{p}$ (generally, for example, $(\hat{q} \otimes \hat{q})_{ij} = q_i q_j$), as follows:

$$\begin{aligned} h_{ij}^{\mathrm{TT}} &= \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} = (h_+ \hat{q} \otimes \hat{q} - h_+ \hat{p} \otimes \hat{p} + h_\times \hat{q} \otimes \hat{p} + h_\times \hat{p} \otimes \hat{q})_{ij} \\ &= h_+ \underbrace{(\hat{q} \otimes \hat{q})_{ij}}_{= q_i q_j} - h_+ \underbrace{(\hat{p} \otimes \hat{p})_{ij}}_{= p_i p_j} + h_\times \underbrace{(\hat{q} \otimes \hat{p})_{ij}}_{= q_i p_j} + h_\times \underbrace{(\hat{p} \otimes \hat{q})_{ij}}_{= p_i q_j} \\ &= h_+ (q_i q_j - p_i p_j) + h_\times (q_i p_j + p_i q_j) \end{aligned}$$

Any metric perturbation h_{ij} in this wave frame basis is then going to take the general form:

$$h_{ij} = h_{ij}^{\mathrm{TT}} + (\text{stuff})_{ij} = h_+(q_i q_j - p_i p_j) + h_\times(q_i p_j + p_i q_j) + (\text{stuff})_{ij}$$

We could now consider **contracting** both sides with, say, $q^i q^j - p^i p^j$. The $(\text{stuff})_{ij}$ contains things like trace terms of the form $\mathrm{Tr}(h_{ij})\delta_{ij}$ or terms proportional to k_i, which are zero by the normalization and transversality of q_i and p_i. Schematically, we would get:

$$\left(q^i q^j - p^i p^j\right)(\text{stuff})_{ij} \propto \left(q^i q^j - p^i p^j\right)(\delta_{ij} + k_i) = \underbrace{q^i q_i}_{=1} - \underbrace{p^i p_i}_{=1} + q^j \underbrace{q^i k_i}_{=0} - p^j \underbrace{p^i k_i}_{=0} = 0$$

We can similarly calculate what contractions of the terms in front of h_+ and $h_\times$ would give:

$$\left(q^i q^j - p^i p^j\right)(q_i q_j - p_i p_j) = \underbrace{q^i q_i}_{=1} \underbrace{q^j q_j}_{=1} - \underbrace{q_i p^i}_{=0} \underbrace{p^j q_j}_{=0} - \underbrace{q^i p_i}_{=0} \underbrace{q^j p_j}_{=0} + \underbrace{p^i p_i}_{=1} \underbrace{p^j p_j}_{=1} = 2$$

$$\left(q^i q^j - p^i p^j\right)(q_i p_j + p_i q_j) = \underbrace{q^i q_i}_{=1} \underbrace{q^j p_j}_{=0} + \underbrace{q^i p_i}_{=0} \underbrace{q^j q_j}_{=1} - \underbrace{p^i q_i}_{=0} \underbrace{p^j p_j}_{=1} - \underbrace{p^i p_i}_{=1} \underbrace{p^j q_j}_{=0} = 0$$

So, if we apply this contraction to a *general* h_{ij}, we would conveniently end up with:

$$\left(q^i q^j - p^i p^j\right) h_{ij} = \left(q^i q^j - p^i p^j\right)\left[h_+(q_i q_j - p_i p_j) + h_\times(q_i p_j + p_i q_j) + (\text{stuff})_{ij}\right] = 2h_+$$

We can therefore extract the **plus-polarization component** directly from h_{ij} as:

$$\boxed{h_+ = \frac{1}{2}\left(q^i q^j - p^i p^j\right) h_{ij}}$$

If we instead were to contract h_{ij} with $q^i p^j + p^i q^j$, we would find:

$$\left(q^i p^j + p^i q^j\right) h_{ij} = \left(q^i p^j + p^i q^j\right)\left[h_+(q_i q_j - p_i p_j) + h_\times(q_i p_j + p_i q_j) + (\text{stuff})_{ij}\right]$$
$$= h_+ \underbrace{\left(q^i p^j + p^i q^j\right)(q_i q_j - p_i p_j)}_{=0} + h_\times \underbrace{\left(q^i p^j + p^i q^j\right)(q_i p_j + p_i q_j)}_{=2} + \underbrace{\left(q^i p^j + p^i q^j\right)(\text{stuff})_{ij}}_{=0} = 2h_\times$$

Then, dividing by 2 on both sides gives us the **cross-polarization component**:

$$\boxed{h_\times = \frac{1}{2}\left(q^i p^j + p^i q^j\right) h_{ij}}$$

Okay, let's take a step back - we have some set of spatial metric components h_{ij}, which can essentially be anything (usually computed from the Einstein field equations). By our framework here, we can *extract* the plus- and cross-polarization components h_+ and $h_\times$ directly from this arbitrary h_{ij} by contracting it with the wave frame basis like above. Once we know h_+ and $h_\times$ (i.e., the two physical degrees of freedom of a gravitational plane wave), we can build the full metric perturbation from them by the formula from earlier:

$$h_{ij}^{\mathrm{TT}} = h_+(q_i q_j - p_i p_j) + h_\times(q_i p_j + p_i q_j) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We could even write the TT-perturbation directly by just plugging the above expressions for h_+ and $h_\times$ into here (note the replacement of the dummy indices i, j with m, n):

$$\begin{aligned} h_{ij}^{\mathrm{TT}} &= \frac{1}{2}\left(q^m q^n - p^m p^n\right) h_{mn}(q_i q_j - p_i p_j) + \frac{1}{2}\left(q^m p^n + p^m q^n\right) h_{mn}(q_i p_j + p_i q_j) \\ &= \frac{1}{2}\left[\left(q^m q^n - p^m p^n\right)(q_i q_j - p_i p_j) + \left(q^m p^n + p^m q^n\right)(q_i p_j + p_i q_j)\right] h_{mn} \end{aligned}$$

You might see this expression in the literature as the **transverse-traceless projector**:

$$\boxed{\Lambda_{ij}^{mn} = \frac{1}{2}\left[\left(q^m q^n - p^m p^n\right)(q_i q_j - p_i p_j) + \left(q^m p^n + p^m q^n\right)(q_i p_j + p_i q_j)\right]}$$

Using this, the transverse-traceless perturbation is then directly $h_{ij}^{\mathrm{TT}} = \Lambda_{ij}^{mn} h_{mn}$. This is really just a different (but equivalent) way of applying the *transverse-traceless gauge* to h_{ij} because the TT-projector satisfies $k^i \Lambda_{ij}^{mn} = 0$ and $\delta^{ij} \Lambda_{ij}^{mn} = 0$ (the TT-gauge conditions).

We've now developed a more or less general framework for extracting the gravitational wave part from any metric perturbation. We'll use this approach later in this lesson.

1. **Define the wave frame**, meaning a basis $\{\hat{q}, \hat{p}, \hat{k}\}$ where the wave propagation direction is $\hat{k} = \vec{k}/k$. There are many valid choices for the polarization basis $\{\hat{q}, \hat{p}\}$.

2. **Extract the plus- and cross-components** from the general metric perturbation h_{ij}:

$$h_+ = \frac{1}{2}\left(q^i q^j - p^i p^j\right) h_{ij}$$
$$h_\times = \frac{1}{2}\left(q^i p^j + p^i q^j\right) h_{ij}$$

3. *If needed,* the full gravitational wave metric can be constructed as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{\mathrm{TT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+h_+ & h_\times & 0 \\ 0 & h_\times & 1-h_+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ with } h_{\mu\nu}^{\mathrm{TT}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

You can also get the TT-metric directly from $h_{ij}^{\mathrm{TT}} = \Lambda_{ij}^{mn} h_{mn}$. Usually, you'll want the h_+ and $h_\times$ components anyway, so the first method may be more practical.

Example: Extracting Transverse-Traceless Gravitational Waves

To put this procedure into more concrete terms, we can look at a simple example. Let's say someone hands us a metric perturbation with spatial components of the form:

$$h_{ij} = A_{ij} \cos(\omega t - kz), \text{ where } A_{ij} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

This describes a "wave-like" perturbation propagating in the z-direction, however, we don't know yet if it actually represents any physical gravitational wave. These are contained in the transverse-traceless part of h_{ij}, and this certainly contains a bunch of additional junk. Our goal now is to extract those physical degrees of freedom from this.

We can first see that the propagation direction is $\hat{k} = \vec{e}_z$, so defining the polarization basis $\{\hat{q}, \hat{p}\} = \{\vec{e}_x, \vec{e}_y\}$ is a very natural choice for this problem. The components of our polarization basis vectors are now simply $q^i = (1, 0, 0)$ and $p^i = (0, 1, 0)$.

We can now extract the plus-polarization mode from our general h_{ij} as:

$$
\begin{aligned}
h_+ &= \frac{1}{2}\left(q^i q^j - p^i p^j\right) h_{ij} \\
&= \frac{1}{2}\Big[\left(q^1 q^1 - p^1 p^1\right) h_{11} + \left(q^1 q^2 - p^1 p^2\right) h_{12} + \left(q^2 q^1 - p^2 p^1\right) h_{21} + \left(q^2 q^2 - p^2 p^2\right) h_{22}\Big] \\
&= \frac{1}{2}(A_{11} - A_{22})\cos(\omega t - kz)
\end{aligned}
$$

Similarly, the cross-polarized mode would end up as:

$$
h_\times = \frac{1}{2}\left(q^i p^j + p^i q^j\right) h_{ij} = A_{12}\cos(\omega t - kz)
$$

So, our general h_{ij} contains a whole bunch of different components, but the actual physical, gravitational wave part of it is given by:

$$
h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \dfrac{A_{11} - A_{22}}{2}\cos(\omega t - kz) & A_{12}\cos(\omega t - kz) & 0 \\ A_{12}\cos(\omega t - kz) & -\dfrac{A_{11} - A_{22}}{2}\cos(\omega t - kz) & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

The general process is simple like this as long as we are able to construct the wave frame $\{\hat{q}, \hat{p}, \hat{k}\}$ in a usable way - that's really what all of this relies on.

5. Sources of Gravitational Waves

So far, we've seen how gravitational waves arise as **vacuum solutions** to the linearized Einstein field equations. Their physical properties, such as their independent transverse polarization modes, are ultimately revealed by imposing the **transverse-traceless gauge**.

The question now is, how are such waves created in the first place? Well, like any gravitational field, gravitational waves are also produced by a matter distribution, so the next step would be to solve the field equations in the presence of a matter source.

The plan for this section is as follows. First, we'll develop some intuition for the fact that gravitational waves are produced by matter configurations with a *time-varying* quadrupole moment. After that, we'll derive the **quadrupole formula** as a solution to the linearized Einstein's equations. It is what allows us to calculate gravitational waves from a source in the **far-field region** and connects the source $T_{\mu\nu}$ to the perturbation $h_{\mu\nu}^{\mathrm{TT}}$.

5.1. How Are Gravitational Waves Produced?

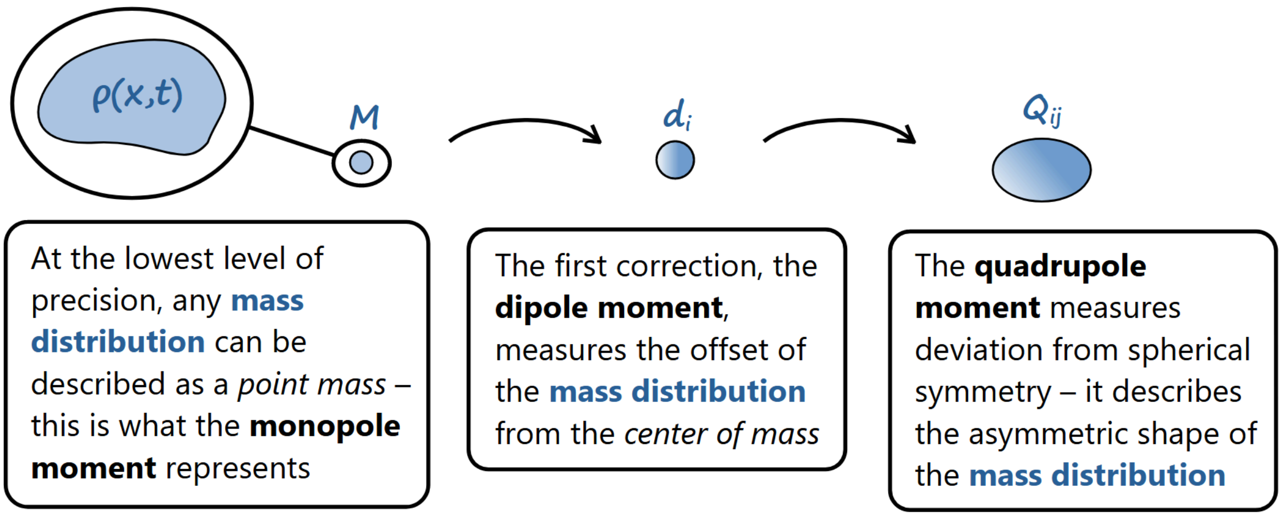
Gravitational waves are created by a *source* with some energy-momentum tensor $T_{\mu\nu}$. For gravitational waves specifically, we often assume the source itself to be *non-relativistic* and contained inside a well-defined region (a so-called *compact source*). These types of sources are mainly described by their **mass distribution**, $\rho(x, t)$, and its time derivatives.

The idea then is that any mass distribution can be characterized by so-called **moments** (the theoretical construct behind this is called a *multipole expansion*). These moments represent higher-order *corrections* to the gravitational field, and each takes into account the (possibly) asymmetric and uneven shape of the mass distribution $\rho(x, t)$:

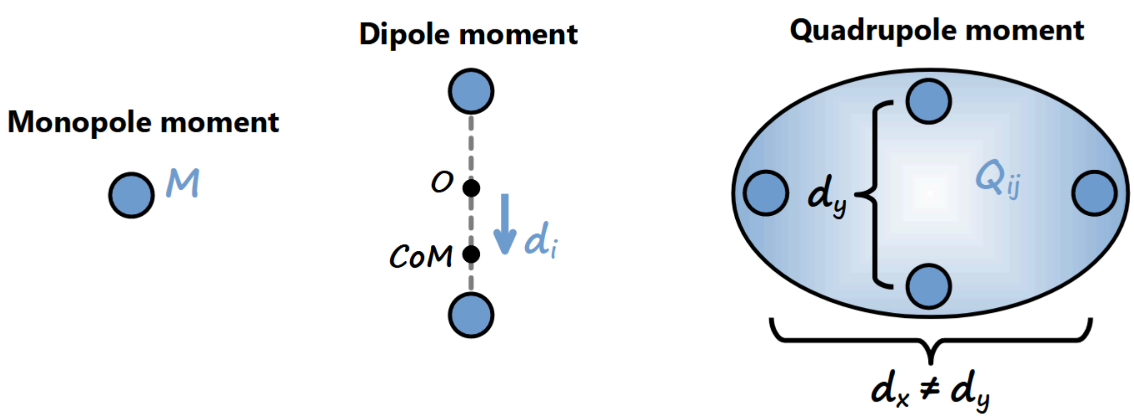
- The **monopole moment**, $M \equiv \iiint \rho dV$, simply measures how much total mass (or rest energy) there is in the source region.
- The **dipole moment**, $d_i \equiv \iiint \rho x_i dV$, is a vector quantity that describes how the mass is distributed along one direction. It's the same thing as the *center of mass*.

- The **quadrupole moment**, $Q_{ij} \equiv \iiint \rho x_i x_j dV$, measures how much the mass distribution deviates from spherical symmetry, so roughly how "elliptical" the distribution is. A perfectly spherical mass distribution - e.g., a point charge - has zero mass quadrupole moment (though this also depends on the choice of origin).

So, if we have any arbitrary clump of matter described by $\rho(x, t)$ inside some region, its gravitational field can be written as a series expansion like "$\text{Field} \sim M + d_i + Q_{ij} + \ldots$ ":



As far as where the names of these come from, they roughly describe the number of point masses needed to construct the corresponding moment. So, *one* point mass would constitute a non-zero *mono*pole moment, *two* masses likewise a non-zero *di*pole moment and *four* masses arranged in the right way a non-zero *quad*rupole moment:

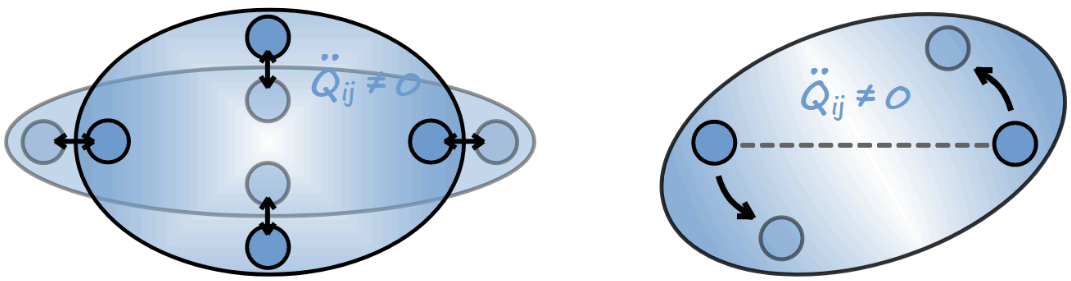


Because the dipole moment of a mass distribution describes the center of mass, we could always make it vanish mathematically by choosing our coordinate origin appropriately (the center-of-mass frame).

Gravitational waves are fundamentally related to the **quadrupole moment** as it captures the "ellipticity" of a mass distribution, which is also how the polarizations of gravitational waves look like. The above are *static* multipole configurations, but to produce waves, we need something that changes with time - a *time-varying* quadrupole moment.

In fact, the connection between quadrupole moments and gravitational waves is much more intuitive than many textbooks make it out to be. The polarization of a gravitational wave is itself basically a **time-varying quadrupolar pattern**. It stretches one direction while simultaneously squeezing another, which generates an *asymmetry* between the two directions. That is precisely what a quadrupole moment characterizes.

What does a *time-varying* quadrupole moment then look like? Well, there are many ways to generate one: for example, we could take the static quadrupole configuration shown above and have the masses oscillate *out of phase*. We could also take a mass dipole and make it spin. In both cases, there is now a quadrupole moment that changes with time:



Notice how the first diagram already looks a lot like a plus-polarized gravitational wave? The second one, in fact, depicts a circularly polarized gravitational wave, which is essentially a rotating ellipse.

So, to produce the plus- and cross-patterns of a gravitational wave, we need a **source whose quadrupole moment changes with time**. Time-varying monopole or dipole moments cannot generate an *asymmetric* polarization pattern with simultaneous stretching and squeezing along *two* directions, which is what a gravitational wave requires. There is, therefore, no monopole or dipole radiation in general relativity.

In the following sections, we will ultimately derive a relationship between the *transverse-traceless* part of the metric perturbation and *second time derivatives* of the quadrupole moment of a mass distribution, $h_{ij}^{\mathrm{TT}} \propto \ddot{Q}_{ij}$, which is the so-called quadrupole formula.

5.2. General Solution of The Linear Field Equations

If the sources of gravitational waves are taken into account, so $T_{\mu\nu} \neq 0$, the linearized Einstein field equations in the Lorenz gauge have the form of a *sourced* wave equation:

$$\partial^\alpha \partial_\alpha h^{\mathrm{TR}}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

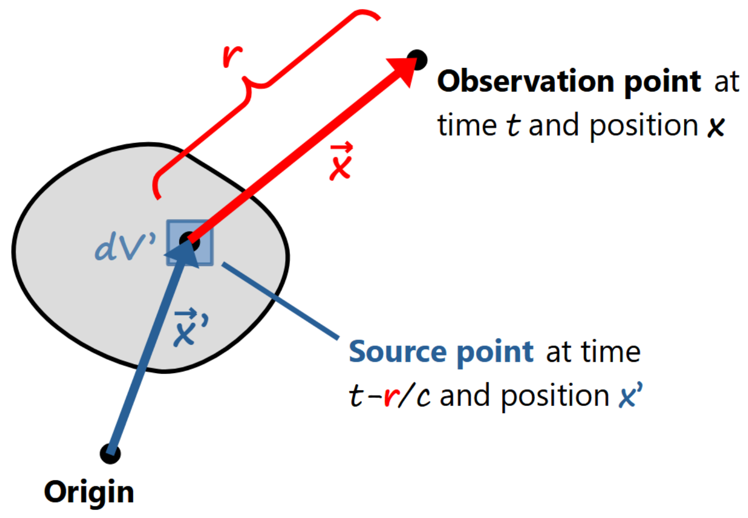
To find out how - or if - a mass distribution produces gravitational waves, we need to solve these equations. The nice thing is that there is a **general solution** to them. At this point, I'll just tell you what this solution is and we can discuss its meaning thereafter:

$$\boxed{h^{\mathrm{TR}}_{\mu\nu}(t, x) = \frac{4G}{c^4} \iiint \frac{T_{\mu\nu}(t - |\vec{x} - \vec{x}'|/c, x')}{|\vec{x} - \vec{x}'|} dV'}$$

The way to "derive" this would be via Green's functions, which we won't cover further here. If you want to learn more, my book Field Theory For The Non-Physicist covers them in greater detail.

Let's go over the key features of this. First, it's given by a **volume integral** over the source region, which is denoted in the integration variable x'. You can think of the source consisting of many point sources, each located at x' and with energy-momentum tensor evaluated at that point, $T_{\mu\nu}(\ldots, x')$. Each source point produces a contribution that scales **inversely with distance** to that source point. This is encapsulated in the factor $1/|\vec{x} - \vec{x}'|$.

The time-dependence appears in the form $t - |\vec{x} - \vec{x}'|/c$. We saw a similar expression for the vacuum wave solutions, $\omega t - kz = \omega(t - z/c)$, which, intuitively, represents a **time delay** due to the finite propagation speed of any gravitational effect. The gravitational field produced by a source point located at x' only "appears" at the observation point x a time $\Delta t = |\vec{x} - \vec{x}'|/c$ later. This is **locality** built directly into the field equation: changes in the gravitational field are *not* instantaneous, but propagate at the (finite!) speed of light c.



One noteworthy point is that this solution relies on the **linearity** of the field equations, meaning the total gravitational field can be obtained by summing up the contributions from each point source. This only works because we're working under *linearized* gravity, not the full *non-linear* general relativity. An explicit solution like this is a very special thing.

Unfortunately, the solution as written above is still quite complicated to do much with analytically. To make progress, we'll apply something called the **far-field approximation** (this term is quite commonly used in antenna theory, though it's appropriate here too). It assumes that our observation point is very far from the source region, which is actually very reasonable as any gravitational waves we're able to detect come from sources many millions of light years away. By any estimate, that can be assumed being "far away".

Mathematically, the far-field approximation sets $|\vec{x}| \gg |\vec{x}'|$, in which case the factor $|\vec{x} - \vec{x}'|$ can be approximated as $|\vec{x} - \vec{x}'| \approx |\vec{x}|$. This basically means that from very far away, the source just looks like a "blob" approximately localized at the origin. Denoting $|\vec{x}| \equiv r$ as the distance from the source ($\approx$ origin) to our observation point, our solution now takes the following form (r can be brought outside as it's not part of the integral):

$$\boxed{h_{\mu\nu}^{\mathrm{TR}}(t, x) \approx \frac{4G}{c^4 r} \iiint T_{\mu\nu}(t - r/c, x')dV'}$$

The far-field approximation also ensures we are in a vacuum far from any matter sources. Therefore, the perturbation $h_{\mu\nu}^{\mathrm{TR}}$ shown here represents a **vacuum solution**.

The idea is that our source, defined by $T_{\mu\nu}$, produces a gravitational field *everywhere*, and that field *far outside* the source can be calculated from what lies inside the source region by the above integral. This is how we connect gravitational waves to what produces them.

About higher-order corrections: while the far-field approximation is valid in most practical scenarios, it's still an approximation, so there are corrections to it. What do these corrections look like? The key idea, as with almost anything in physics, is to *Taylor-expand*. The quantity $|\vec{x} - \vec{x}'|$ is a vector length, which can be calculated as:

$$|\vec{x} - \vec{x}'| = \sqrt{(\vec{x} - \vec{x}') \cdot (\vec{x} - \vec{x}')} = \sqrt{r^2 - 2\vec{x} \cdot \vec{x}' + |\vec{x}'|^2} = r\sqrt{1 - \frac{2\vec{x} \cdot \vec{x}'}{r^2} + \frac{|\vec{x}'|^2}{r^2}}$$

Here, $|\vec{x}|^2 = \vec{x} \cdot \vec{x} = r^2$ and $|\vec{x}'|^2 = \vec{x}' \cdot \vec{x}'$ are just the usual definitions of vector length.

Now, the most significant correction comes from the term $2\vec{x} \cdot \vec{x}' / r^2$, as this is on the order of $\sim |\vec{x}'| / r$. The other term, $|\vec{x}'|^2 / r^2$ is *tiny* in comparison when r is even somewhat large. So, we can drop this term and write:

$$|\vec{x} - \vec{x}'| \approx r\sqrt{1 - \frac{2\vec{x} \cdot \vec{x}'}{r^2}}$$

Before any further Taylor-expansions, note that the factor $|\vec{x} - \vec{x}'|$ appears in two different places in our source integral: in the denominator in the integral and also in the time-dependence of $T_{\mu\nu}$. This is important because the series expansion for these scale differently with r, namely by $(1 + x)^{-1/2} \approx 1 - x/2$ and $(1 + x)^{1/2} \approx 1 + x/2$:

$$|\vec{x} - \vec{x}'| = r\sqrt{1 - \frac{2\vec{x} \cdot \vec{x}'}{r^2}} \approx r\left(1 - \frac{1}{2}\frac{2\vec{x} \cdot \vec{x}'}{r^2}\right) = r - \frac{\vec{x} \cdot \vec{x}'}{r}$$

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{r\sqrt{1 - \frac{2\vec{x} \cdot \vec{x}'}{r^2}}} \approx \frac{1}{r}\left(1 + \frac{1}{2}\frac{2\vec{x} \cdot \vec{x}'}{r^2}\right) = \frac{1}{r} + \frac{\vec{x} \cdot \vec{x}'}{r^3}$$

We can see that the first higher-order correction in $1/|\vec{x}-\vec{x}'|$ is on the order $\sim r^{-3}$, which again is tiny compared to the r^{-1}-term in $|\vec{x}-\vec{x}'|$. A good approximation that incorporates the next most important correction is given by taking $1/|\vec{x}-\vec{x}'| \approx 1/r$, but $|\vec{x}-\vec{x}'| \approx r - \vec{x}\cdot\vec{x}'/r$ in the "time delay" term:

$$h_{\mu\nu}^{\mathrm{TR}} = \frac{4G}{c^4}\iiint \frac{T_{\mu\nu}(t-|\vec{x}-\vec{x}'|/c, x')}{|\vec{x}-\vec{x}'|}dV' \approx \frac{4G}{c^4 r}\iiint T_{\mu\nu}\left(t-\frac{r}{c}+\frac{\vec{x}\cdot\vec{x}'}{rc}, x'\right)dV'$$

We're not done yet. The term $\vec{x}\cdot\vec{x}'/rc$ is still generally small compared to r/c, so we can do another Taylor expansion. If we have some function $f(z)$ and we add a small number Δz to its input argument, then to first order, the function can be expanded as:

$$f(z+\Delta z) \approx f(z) + \frac{df}{dz}\Delta z$$

For this energy-momentum tensor, we replace $z \longrightarrow t - r/c$, $\Delta z \longrightarrow \vec{x}\cdot\vec{x}'/rc$ and $d/dz \longrightarrow \partial/\partial t$, such that:

$$T_{\mu\nu}\left(t-\frac{r}{c}+\frac{\vec{x}\cdot\vec{x}'}{rc}, x'\right) \approx T_{\mu\nu}\left(t-\frac{r}{c}, x'\right) + \frac{\partial T_{\mu\nu}}{\partial t}\bigg|_{t-r/c}\frac{\vec{x}\cdot\vec{x}'}{rc}$$

We can suppress the arguments of these for now, but just note that everything should be evaluated at the spacetime point $(t-r/c, x')$. This gives us then:

$$\begin{aligned} h_{\mu\nu}^{\mathrm{TR}} &\approx \frac{4G}{c^4 r}\iiint\left(T_{\mu\nu} + \frac{\partial T_{\mu\nu}}{\partial t}\frac{\vec{x}\cdot\vec{x}'}{rc}\right)dV' \\ &= \frac{4G}{c^4 r}\iiint T_{\mu\nu}dV' + \frac{4G}{c^5 r^2}\frac{d}{dt}\iiint T_{\mu\nu}\vec{x}\cdot\vec{x}'dV' \end{aligned}$$

The second term here represents the first higher-order correction to the metric. The integral has a factor $\vec{x}\cdot\vec{x}'$, which roughly accounts for **shape variations** across different parts of the source. These each contribute to the gravitational field at slightly different times, and the total delay effect appears as a time derivative.

5.3. The Quadrupole Formula

We now have the general far-field solution to the linearized field equations expressed as an integral over the energy-momentum tensor. In principle, we can use it to calculate the gravitational field produced by any source object. Still, to actually calculate the metric from it, we would have to know all components of the energy-momentum tensor:

$$h_{00}^{\mathrm{TR}} = \frac{4G}{c^4 r}\iiint T_{00}dV, \quad h_{0i}^{\mathrm{TR}} = \frac{4G}{c^4 r}\iiint T_{0i}dV, \quad h_{ij}^{\mathrm{TR}} = \frac{4G}{c^4 r}\iiint T_{ij}dV$$

Note: we've replaced $dV' \rightarrow dV$ *(i.e.* d^3x' *with* d^3x*) to make the notation a bit less cluttered.*

In practical scenarios, we often don't have this much information about the source. For example, we might only know its mass (T_{00}) through astronomical observations but not much about its internal structure like internal pressure, stresses or momentum flux (T_{ij}).

Luckily, there exists a useful result known as the **tensor virial theorem**, which states that for *isolated* sources, integrals of the momentum density and flux components, T_{0i} and T_{ij}, can be calculated from *time derivatives* of integrals involving T_{00} and coordinates x_i:

$$\boxed{\begin{aligned}\iiint T_{0i}dV &= -\frac{1}{c}\frac{d}{dt}\iiint T_{00}x_i dV \\ \iiint T_{ij}dV &= \frac{1}{2c^2}\frac{d^2}{dt^2}\iiint T_{00}x_i x_j dV\end{aligned}}$$

Derivation of The Tensor Virial Theorem

Deriving the tensor virial theorem is not too difficult but it does rely on a few tricks that seem a bit obscure at first. The starting point is to consider the divergence of the quantity $T_{km}x_i x_j$, which can be calculated with use of the product rule:

$$\partial^m(T_{km}x_i x_j) = \partial^m T_{km}x_i x_j + T_{km}\partial^m(x_i x_j) = \partial^m T_{km}x_i x_j + T_{km}\partial^m x_i x_j + T_{km}x_i\partial^m x_j$$

These partial derivatives of coordinates give us Kronecker deltas, by definition:

$$\partial^m x_i = \frac{\partial x_i}{\partial x_m} = \delta_i^m$$

Our expression above then becomes:

$$\begin{aligned}\partial^m (T_{km} x_i x_j) &= \partial^m T_{km} x_i x_j + T_{km} \partial^m x_i x_j + T_{km} x_i \partial^m x_j \\ &= \partial^m T_{km} x_i x_j + T_{km} \delta_i^m x_j + T_{km} x_i \delta_j^m \\ &= \partial^m T_{km} x_i x_j + T_{ki} x_j + T_{kj} x_i\end{aligned}$$

Next, we'll take another divergence of this, which by similar steps gives:

$$\begin{aligned}\partial^k \partial^m (T_{km} x_i x_j) &= \partial^k \left(\partial^m T_{km} x_i x_j\right) + \partial^k (T_{ki} x_j) + \partial^k (T_{kj} x_i) \\ &= \partial^k \partial^m T_{km} x_i x_j + \partial^m T_{km} \left(\partial^k x_i x_j + x_i \partial^k x_j\right) + \partial^k T_{ki} x_j + T_{ki} \partial^k x_j \\ &\quad + \partial^k T_{kj} x_i + T_{kj} \partial^k x_i \\ &= \partial^k \partial^m T_{km} x_i x_j + \partial^m T_{km} \left(\delta_i^k x_j + x_i \delta_j^k\right) + \partial^k T_{ki} x_j + T_{ki} \delta_j^k \\ &\quad + \partial^k T_{kj} x_i + T_{kj} \delta_i^k \\ &= \partial^k \partial^m T_{km} x_i x_j + \partial^m T_{im} x_j + \partial^m T_{jm} x_i + \partial^k T_{ki} x_j + T_{ji} + \partial^k T_{kj} x_i + T_{ij}\end{aligned}$$

Because the energy-momentum tensor is symmetric, $T_{ji} = T_{ij}$. We can also relabel the dummy index m to k on the second and third terms, so that these terms combine with the fourth and the sixth terms. We're then left with:

$$\partial^k \partial^m (T_{km} x_i x_j) = \partial^k \partial^m T_{km} x_i x_j + 2\partial^k T_{ki} x_j + 2\partial^k T_{kj} x_i + 2T_{ij}$$

Now, one more product rule trick - the second and third terms can be written as:

$$\begin{aligned}\partial^k T_{ki} x_j &= \partial^k (T_{ki} x_j) - T_{ji} \\ \partial^k T_{kj} x_i &= \partial^k (T_{kj} x_i) - T_{ij}\end{aligned}$$

You can verify these hold by writing out the derivatives on the right.

Using these, we can write the quantity $\partial^k\partial^m(T_{km}x_ix_j)$ in the form:

$$\begin{aligned}\partial^k\partial^m(T_{km}x_ix_j) &= \partial^k\partial^m T_{km}x_ix_j + 2\left(\partial^k(T_{ki}x_j) - T_{ji}\right) + 2\left(\partial^k(T_{kj}x_i) - T_{ij}\right) + 2T_{ij} \\ &= \partial^k\partial^m T_{km}x_ix_j + 2\partial^k(T_{ki}x_j) + 2\partial^k(T_{kj}x_i) - 2T_{ij} \\ &= \partial^k\partial^m T_{km}x_ix_j + 2\partial^k(T_{ki}x_j + T_{kj}x_i) - 2T_{ij}\end{aligned}$$

The point of all of this is to rewrite the volume integral of T_{ij}, so we can solve this to get an alternative expression for T_{ij} and then take the volume integral on both sides:

$$\iiint T_{ij}dV = \frac{1}{2}\iiint \partial^k\partial^m T_{km}x_ix_jdV + \iiint \partial^k\left(T_{ki}x_j + T_{kj}x_i - \frac{1}{2}\partial^m(T_{km}x_ix_j)\right)dV$$

The second term on the right is a *total divergence*. The **divergence theorem** states that the volume integral of a divergence can be written as a surface integral over the **boundary**, so if we denote $A_{kij} = T_{ki}x_j + T_{kj}x_i - \partial^m(T_{km}x_ix_j)/2$, the second term is:

$$\iiint \partial^k A_{kij}dV = \oiint A_{kij}n^k dS$$

Our volume integral here is taken over the source region, and we're assuming that whatever that source region is, all energy and momentum is contained inside it. If nothing leaves the source region, the energy-momentum tensor should itself go to zero at the boundary (meaning $A_{kij} \propto T_{ki} \rightarrow 0$ as we approach the boundary of our source region). Therefore, this surface term should be zero and we're left with:

$$\iiint T_{ij}dV = \frac{1}{2}\iiint \partial^k\partial^m T_{km}x_ix_jdV$$

There is one last step we need to do, and that is to use the conservation of the energy-momentum tensor, $\partial^\mu T_{\mu\nu} = 0$. This can be split into two equations, one for $\nu = 0$ and another for $\nu = i$ (which itself is really three equations). For both equations, we can write out the summations over the dummy index μ in the following form:

$$\begin{cases}\partial^\mu T_{\mu 0} = \partial^0 T_{00} + \partial^i T_{i0} = 0 \\ \partial^\mu T_{\mu i} = \partial^0 T_{0i} + \partial^k T_{ki} = 0\end{cases} \Rightarrow \begin{cases}\partial^0 T_{00} = -\partial^i T_{i0} \\ \partial^0 T_{0i} = -\partial^k T_{ki}\end{cases}$$

Taking the ∂^i-derivative of the second equation and inserting it into the first equation:

$$\begin{aligned}
&\partial^i \partial^0 T_{0i} = -\partial^i \partial^k T_{ki} \\
&\Rightarrow \quad \partial^0 \underbrace{\partial^i T_{0i}}_{-\partial^0 T_{00}} = -\partial^i \partial^k T_{ki} \\
&\Rightarrow \quad \partial^0 \partial^0 T_{00} = \partial^i \partial^k T_{ki}
\end{aligned}$$

So, energy-momentum conservation allows us to rewrite the "double-divergence" of T_{km} in terms of the second time derivative of T_{00}. Our relation above then becomes:

$$\iiint T_{ij} dV = \frac{1}{2} \iiint \partial^k \partial^m T_{km} x_i x_j dV = \frac{1}{2} \iiint \partial^0 \partial^0 T_{00} x_i x_j dV$$

The last thing to do is to write $\partial^0 \partial^0 = \partial_t^2 / c^2$ and move the partial derivatives outside the integral to become total derivatives (this is allowed because we assume the integration variables themselves do not depend on time). The final result is:

$$\iiint T_{ij} dV = \frac{1}{2c^2} \frac{d^2}{dt^2} \iiint T_{00} x_i x_j dV$$

Similarly, we can use energy-momentum conservation on $\partial^j (T_{0j} x_i) = \partial^j T_{0j} x_i + T_{0i}$ to write $\partial^j T_{j0} = -\partial^0 T_{00} = c^{-1} \partial_t T_{00}$, and then integrate both sides:

$$\begin{aligned}
&\partial^j (T_{0j} x_i) = \partial^j T_{0j} x_i + T_{0i} \\
&\Rightarrow \quad \iiint \partial^j (T_{0j} x_i) dV = \iiint \frac{1}{c} \partial_t T_{00} x_i dV + \iiint T_{0i} dV \\
&\Rightarrow \quad \oiint T_{0j} x_i n^j dS = \frac{1}{c} \frac{d}{dt} \iiint T_{00} x_i dV + \iiint T_{0i} dV
\end{aligned}$$

The surface integral on the left (which comes from the divergence theorem again) should vanish by the same logic as earlier, so that:

$$\iiint T_{0i} dV = -\frac{1}{c} \frac{d}{dt} \iiint T_{00} x_i dV$$

The useful thing about the tensor virial theorem relations is that they allow us to write integrals over the source region **only in terms of T_{00}**, which is arguably the simplest energy-momentum tensor component. Consequently, we can then write the components $h_{\mu\nu}^{\mathrm{TR}}$ of our solution to the linearized field equations also completely in terms of T_{00}:

$$h_{00}^{\mathrm{TR}} = \frac{4G}{c^4 r}\iiint T_{00} dV, \quad h_{0i}^{\mathrm{TR}} = -\frac{4G}{c^5 r}\frac{d}{dt}\iiint T_{00} x_i dV, \quad h_{ij}^{\mathrm{TR}} = \frac{2G}{c^6 r}\frac{d^2}{dt^2}\iiint T_{00} x_i x_j dV$$

For non-relativistic (weak and slowly moving) matter, we can take $T_{00} \approx \rho c^2$. This gives the metric components in terms of the *monopole*, *dipole* and *quadrupole moments*:

$$\begin{cases} h_{00}^{\mathrm{TR}} = \dfrac{4G}{c^4 r}\displaystyle\iiint \rho c^2 dV \equiv \dfrac{4GM}{c^2 r} \\ h_{0i}^{\mathrm{TR}} = -\dfrac{4G}{c^5 r}\dfrac{d}{dt}\displaystyle\iiint \rho c^2 x_i dV \equiv -\dfrac{4G}{c^3 r}\dot{d}_i \\ h_{ij}^{\mathrm{TR}} = \dfrac{2G}{c^6 r}\dfrac{d^2}{dt^2}\displaystyle\iiint \rho c^2 x_i x_j dV \equiv \dfrac{2G}{c^4 r}\ddot{Q}_{ij} \end{cases}$$

The dots here are short-hand for time derivatives, meaning $\dot{d}_i = dd_i / dt$ and $\ddot{Q}_{ij} = d^2 Q_{ij} / dt^2$.

Remember what we discussed earlier - only a **time-varying quadrupole moment** radiates gravitational waves. The monopole moment M (= total mass) should be constant in a closed system and the dipole moment d_i can be made to vanish by an appropriate reference frame choice, so neither h_{00}^{TR} or h_{0i}^{TR} contain any physical gravitational waves.

If gravitational waves are all we care about, we should next apply the **transverse-traceless gauge**, which keeps the *radiative* components but sets everything else to zero. Our solution here is in the far-field, so the transverse-traceless gauge should be valid. In that case, applying it sets $h_{0\mu}^{\mathrm{TT}} = 0$, and our solution becomes completely specified by h_{ij}^{TT}.

How do now we get the *transverse-traceless* part h_{ij}^{TT} from our solution written in terms of the *trace-reversed* perturbation? We can do so by the approach developed earlier: first, we set up a **wave frame** $\{\hat{q}, \hat{p}, \hat{k}\}$, which also defines the transverse-traceless projector Λ_{ij}^{mn}.

Then, by acting with the TT-projector on the *trace-reversed perturbation*, we'll get the *transverse-traceless perturbation* $h_{ij}^{\mathrm{TT}} = \Lambda_{ij}^{mn} h_{mn}^{\mathrm{TR}}$. This gives us the **quadrupole formula**:

$$\boxed{h_{ij}^{\mathrm{TT}} = \frac{2G}{c^4 r}\Lambda_{ij}^{mn}\ddot{Q}_{ij}(t - r/c)}$$

, where $Q_{ij} = \iiint \rho x_i x_j dV$.

Reminder: $\Lambda_{ij}^{mn} = 1/2\left(q^m q^n - p^m p^n\right)(q_i q_j - p_i p_j) + 1/2\left(q^m p^n + p^m q^n\right)(q_i p_j + p_i q_j)$.

This very well-known formula allows us to calculate the gravitational waves produced from any source in the far-field region. It puts our earlier reasoning of why a time-varying quadrupole moment radiates into mathematical terms. If we know the mass distribution ρ of our source region, we can calculate how it radiates out gravitational waves!

In practice, we'll often want the **two polarization components** as well. The quadrupole formula can be equivalently written as *two* equations for the plus- and cross-components by using $h_+ = 1/2\left(q^i q^j - p^i p^j\right) h_{ij}^{\mathrm{TR}}$ and $h_\times = 1/2\left(q^i p^j + p^i q^j\right) h_{ij}^{\mathrm{TR}}$ from earlier:

$$\boxed{h_+ = \frac{G}{c^4 r}\left(q^i q^j - p^i p^j\right)\ddot{Q}_{ij}(t - r/c)}$$

$$\boxed{h_\times = \frac{G}{c^4 r}\left(q^i p^j + p^i q^j\right)\ddot{Q}_{ij}(t - r/c)}$$

These expressions describe how any source configuration radiates the two characteristic polarization modes, h_+ and $h_\times$. They allow us to directly calculate these (with respect to our choice of the polarization basis $\{\hat{q}, \hat{p}\}$) from essentially just the mass distribution.

How To Calculate Gravitational Waves From a Source

1. **Write down the mass density function**, $\rho = \rho(t, x)$. For a source consisting of discrete masses (e.g., binary stars), you'll need the positions of each mass.

2. **Calculate the quadrupole moment** from the volume integral:

$$Q_{ij} = \iiint \rho x_i x_j dV$$

3. **Take the second time derivative of the quadrupole moment and evaluate it at the retarded time**, $t - r/c$, which gives you $\ddot{Q}_{ij}(t - r/c)$. This is the step that gives the "wave-like" behaviour, $\ddot{Q}_{ij}(t - r/c) \sim f(\omega t - kr)$.

4. **Define a wave frame**, meaning a basis $\{\hat{q}, \hat{p}, \hat{k}\}$ based on the wave propagation direction $\hat{k} = \vec{k}/k$. In the far-field region, the propagation direction $\hat{k}$ is always radially outward from the origin.

5. **Calculate the characteristic polarization components** of the produced waves:

$$h_+ = \frac{G}{c^4 r}\left(q^i q^j - p^i p^j\right)\ddot{Q}_{ij}(t - r/c)$$
$$h_\times = \frac{G}{c^4 r}\left(q^i p^j + p^i q^j\right)\ddot{Q}_{ij}(t - r/c)$$

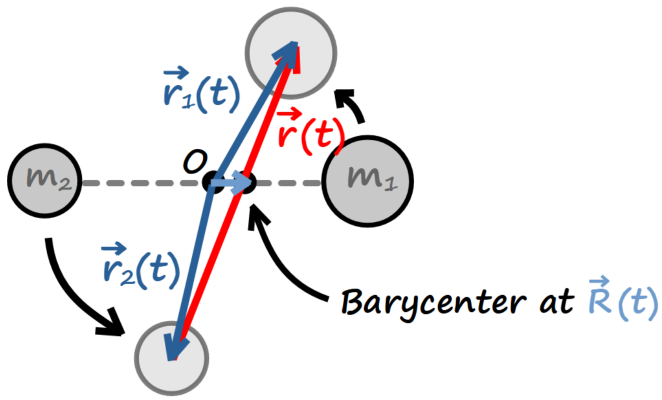
Equivalently, you can also use the full **quadrupole formula**:

$$h_{ij}^{\mathrm{TT}} = \frac{2G}{c^4 r}\Lambda_{ij}^{mn}\ddot{Q}_{ij}(t - r/c)$$

5.4. Example: Binary Star Systems

To finish our discussion of gravitational waves, we'll work out a full example. One of the simplest systems that can produce gravitational waves is a **binary mass system**: two masses (e.g., stars) orbiting each other around a common center of mass. Such a system has a time-varying quadrupole moment, and can therefore radiate gravitational waves.

For this example, we'll take two masses m_1 and m_2 both in **circular orbits**. We can place this system in the xy-plane and denote the positions of the masses $\vec{r}_1(t)$ and $\vec{r}_2(t)$. The masses orbit a common center of mass, or *barycenter* (closer to the larger mass), the position of which we'll denote $\vec{R}(t)$. The separation of the masses is $\vec{r}(t) = \vec{r}_1(t) - \vec{r}_2(t)$:



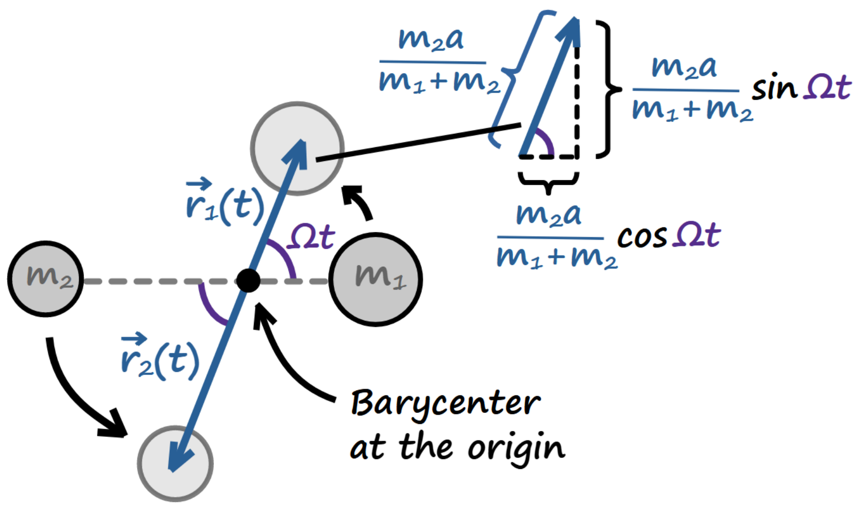
This problem is basically a special case of the more general **two-body problem**. We'll need just one result from theory of the two-body problem, which is that the positions of the two masses can be written in terms of the center of mass position $\vec{R}(t)$ - which is equal to the dipole moment, in fact - and the separation between the bodies, $\vec{r}(t)$, as:

$$\begin{cases} \vec{r}_1(t) = \vec{R}(t) + \dfrac{m_2}{M}\vec{r}(t) \\ \vec{r}_2(t) = \vec{R}(t) - \dfrac{m_1}{M}\vec{r}(t) \end{cases}$$, where $M = m_1 + m_2$ is the total mass.

These relations are completely general and don't require any assumptions about circular orbits. For our problem, however, we can choose to work in the so-called **center-of-mass frame** in which the center of mass is always at the origin, so $\vec{R}(t) = 0$.

If the distance between the masses is a (constant for circular orbits), we can write the separation vector as a function of time as $\vec{r}(t) = (a\cos\Omega t, a\sin\Omega t, 0)$, where Ω is the orbital angular frequency $\Omega = 2\pi / T$. The positions of the masses in this frame are then:

$$\begin{cases} \vec{r}_1(t) = \dfrac{m_2}{M}\vec{r}(t) = \dfrac{m_2 a}{M}(\cos\Omega t,\ \sin\Omega t, 0) \\ \vec{r}_2(t) = -\dfrac{m_1}{M}\vec{r}(t) = -\dfrac{m_1 a}{M}(\cos\Omega t,\ \sin\Omega t, 0) \end{cases}$$



Okay, we have the positions of both masses as functions of time now. The next step is to write down the mass density function of the system, which is given by delta functions:

$$\rho(t, x) = m_1\delta^3(\vec{x} - \vec{r}_1(t)) + m_2\delta^3(\vec{x} - \vec{r}_2(t))$$

The mass density should be zero everywhere except at the positions of the masses where it is technically infinite. Explicitly, $\delta^3(\vec{x} - \vec{r}_1(t)) = \delta(x - m_2 a / M\cos\Omega t)\delta(y - m_2 a / M\sin\Omega t)\delta(z)$.

Next, let's calculate the components of the quadrupole moment, $Q_{ij} = \iiint \rho x_i x_j dV$. We have the coordinates $x_i = (x_1, x_2, x_3) = (x, y, z)$, so the Q_{11}-component would be:

$$Q_{11} = \iiint \rho x_1^2 dV = m_1\iiint x^2\delta^3(\vec{x} - \vec{r}_1(t))dV + m_2\iiint x^2\delta^3(\vec{x} - \vec{r}_2(t))dV$$

When integrating these delta functions, they essentially just pick out the values $x = m_2 a / M \cos \Omega t$ and $x = -m_1 a / M \cos \Omega t$, so the volume integrals become:

$$Q_{11} = m_1 \left(\frac{m_2 a}{M} \cos \Omega t \right)^2 + m_2 \left(-\frac{m_1 a}{M} \cos \Omega t \right)^2 = \frac{m_1 m_2}{M^2} \underbrace{\left(m_2 + m_1 \right)}_{=M} a^2 \cos^2 \Omega t \equiv \mu a^2 \cos^2 \Omega t$$

*We've defined $\mu = m_1 m_2 / M$ as the **reduced mass**. It's a common parameter in orbital mechanics.*

Here, $\mu = m_1 m_2 / M$ is the so-called **reduced mass**. For the other components of Q_{ij}:

$$\begin{aligned} Q_{12} &= \iiint \rho x_1 x_2 dV = m_1 \iiint xy \delta^3(\vec{x} - \vec{r}_1(t)) dV + m_2 \iiint xy \delta^3(\vec{x} - \vec{r}_2(t)) dV \\ &= m_1 \frac{m_2 a}{M} \cos \Omega t \frac{m_2 a}{M} \sin \Omega t + m_2 \left(-\frac{m_1 a}{M} \cos \Omega t \right) \left(-\frac{m_1 a}{M} \sin \Omega t \right) = \mu a^2 \cos \Omega t \sin \Omega t \\ Q_{22} &= \iiint \rho x_2^2 dV = m_1 \iiint y^2 \delta^3(\vec{x} - \vec{r}_1(t)) dV + m_2 \iiint y^2 \delta^3(\vec{x} - \vec{r}_2(t)) dV \\ &= m_1 \left(\frac{m_2 a}{M} \sin \Omega t \right)^2 + m_2 \left(-\frac{m_1 a}{M} \sin \Omega t \right)^2 = \mu a^2 \sin^2 \Omega t \\ Q_{13} &= Q_{23} = Q_{33} = 0 \end{aligned}$$

For the quadrupole formula, we also need to take second time derivatives of these:

$$\begin{aligned} \ddot{Q}_{11} &= \mu a^2 \frac{d^2}{dt^2} \cos^2 \Omega t = -2 \mu a^2 \Omega^2 \left(\cos^2 \Omega t - \sin^2 \Omega t \right) \\ \ddot{Q}_{12} &= \mu a^2 \frac{d^2}{dt^2} (\cos \Omega t \sin \Omega t) = -4 \mu a^2 \Omega^2 \cos \Omega t \sin \Omega t \\ \ddot{Q}_{22} &= \mu a^2 \frac{d^2}{dt^2} \sin^2 \Omega t = 2 \mu a^2 \Omega^2 \left(\cos^2 \Omega t - \sin^2 \Omega t \right) \end{aligned}$$

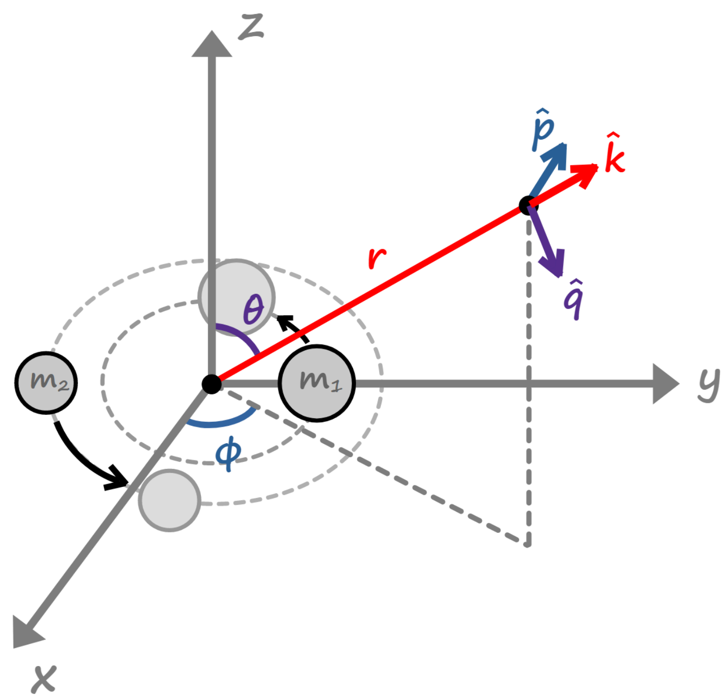
These can be simplified using the trigonometric identities $\cos^2 \Omega t - \sin^2 \Omega t = \cos 2 \Omega t$ and $\cos \Omega t \sin \Omega t = {}^1/_2 \sin 2 \Omega t$, which give:

$$\ddot{Q}_{11} = -2 \mu a^2 \Omega^2 \cos 2 \Omega t \qquad \ddot{Q}_{12} = -2 \mu a^2 \Omega^2 \sin 2 \Omega t \qquad \ddot{Q}_{22} = 2 \mu a^2 \Omega^2 \cos 2 \Omega t$$

For last, we should evaluate these at $t - r/c$ like in the quadrupole formula. This gives us, essentially, wave-like expressions that have a wavenumber $k = \omega / c \equiv 2\Omega / c$:

$$\ddot{Q}_{11}(t - r/c) = -2\mu a^2\Omega^2 \cos(2\Omega(t - r/c)) = -2\mu a^2\Omega^2 \cos(2\Omega t - kr)$$
$$\ddot{Q}_{12}(t - r/c) = -2\mu a^2\Omega^2 \sin(2\Omega(t - r/c)) = -2\mu a^2\Omega^2 \sin(2\Omega t - kr)$$
$$\ddot{Q}_{22}(t - r/c) = 2\mu a^2\Omega^2 \cos(2\Omega(t - r/c)) = 2\mu a^2\Omega^2 \cos(2\Omega t - kr)$$
$$\ddot{Q}_{13} = \ddot{Q}_{23} = \ddot{Q}_{33} = 0$$

To apply our quadrupole formulas and calculate our polarization components, we should choose a **wave frame**. Our gravitational waves are all propagating in the r-direction ($\hat{k} = \hat{e}_r$), so a natural choice would be to choose the **polarization basis** $\{\hat{q}, \hat{p}\}$ as the *normalized* angular basis vectors in spherical coordinates, $\hat{q} = \hat{e}_\theta$ and $\hat{p} = \hat{e}_\varphi$:



The Cartesian component representations of $\hat{q}$ and $\hat{p}$ in terms of r, θ and φ are then:

$$\hat{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} \qquad \hat{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

We now have everything we need to calculate the the plus- and cross-polarization components! These turn out to be, from the quadrupole formulas (we're skipping a bunch of the trigonometric algebra here - you are free to fill in the details yourself, and if you do, you may need the formulas $\cos^2\varphi - \sin^2\varphi = \cos 2\varphi$ and $2\cos\varphi\sin\varphi = \sin 2\varphi$):

$$\begin{aligned} h_+ &= \frac{G}{c^4 r}\left(q^i q^j - p^i p^j\right)\ddot{Q}_{ij}(t - r/c) \\ &= \frac{G}{c^4 r}\left(q^1 q^1 - p^1 p^1\right)\ddot{Q}_{11} + \frac{2G}{c^4 r}\left(q^1 q^2 - p^1 p^2\right)\ddot{Q}_{12} + \frac{G}{c^4 r}\left(q^2 q^2 - p^2 p^2\right)\ddot{Q}_{22} \\ &= -\frac{2G\mu a^2\Omega^2}{c^4 r}\left(1 + \cos^2\theta\right)[\cos 2\varphi \cos(2\Omega t - kr) + \sin 2\varphi \sin(2\Omega t - kr)] \end{aligned}$$

$$\begin{aligned} h_\times &= \frac{G}{c^4 r}\left(q^i p^j + p^i q^j\right)\ddot{Q}_{ij}(t - r/c) \\ &= \frac{G}{c^4 r}\left(q^1 p^1 + p^1 q^1\right)\ddot{Q}_{11} + \frac{2G}{c^6 r}\left(q^1 p^2 + p^1 q^2\right)\ddot{Q}_{12} + \frac{G}{c^6 r}\left(q^2 p^2 + p^2 q^2\right)\ddot{Q}_{22} \\ &= \frac{4G\mu a^2\Omega^2}{c^4 r}\cos\theta[\sin 2\varphi \cos(2\Omega t - kr) - \cos 2\varphi \sin(2\Omega t - kr)] \end{aligned}$$

These rather long trigonometric expressions have the form of trigonometric sum formulas, namely, $\cos 2\varphi \cos(2\Omega t - kr) + \sin 2\varphi \sin(2\Omega t - kr) = \cos(2\Omega t - kr - 2\varphi)$ and $\sin 2\varphi \cos(2\Omega t - kr) - \cos 2\varphi \sin(2\Omega t - kr) = -\sin(2\Omega t - kr - 2\varphi)$). Therefore:

$$\boxed{h_+ = -\frac{2G\mu a^2\Omega^2}{c^4}\frac{1 + \cos^2\theta}{r}\cos(2\Omega t - kr - 2\varphi)}$$

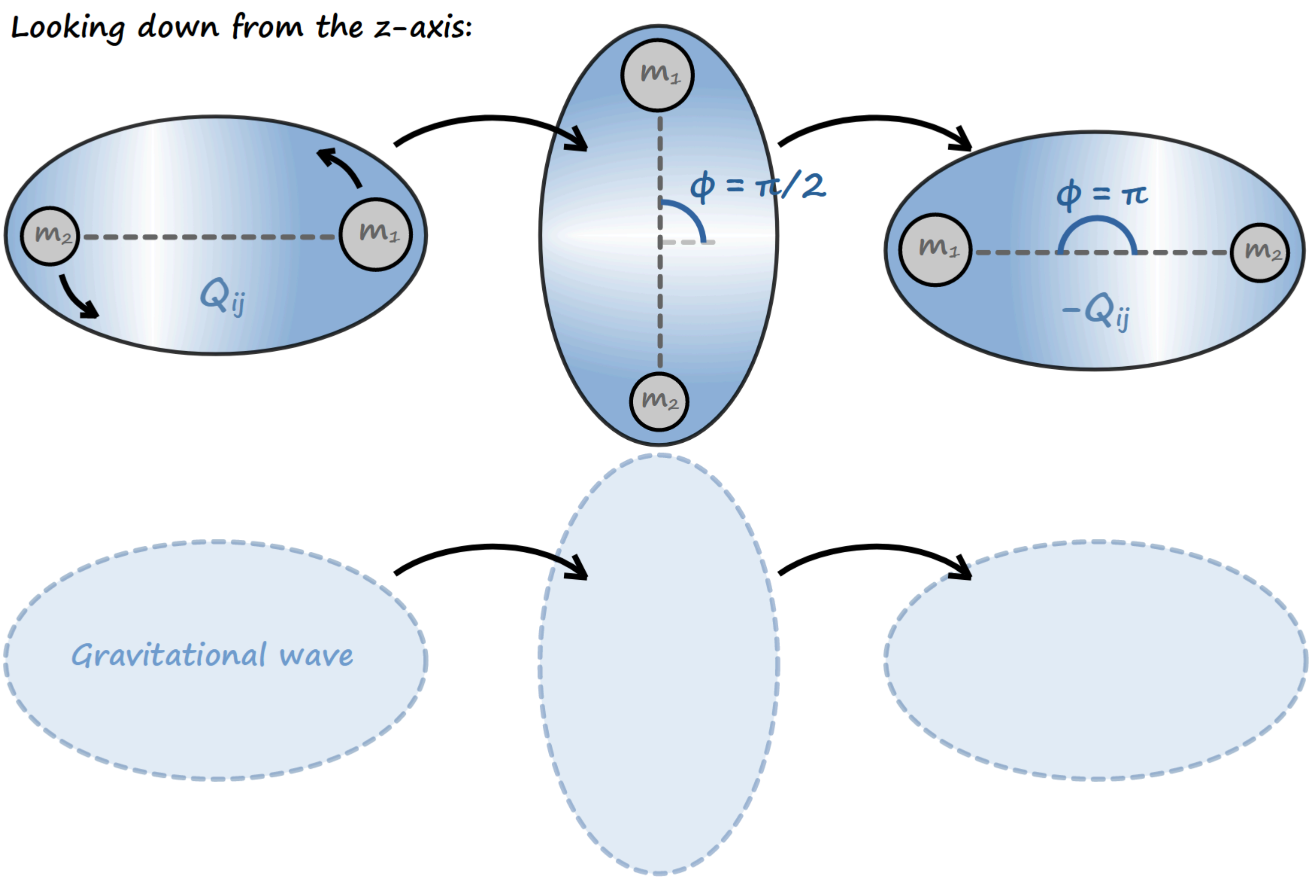
$$\boxed{h_\times = -\frac{4G\mu a^2\Omega^2}{c^4}\frac{\cos\theta}{r}\sin(2\Omega t - kr - 2\varphi)}$$

We could also write the full metric perturbation in the form of a 4×4 matrix:

$$h_{\mu\nu}^{\mathrm{TT}} = -\frac{2G\mu a^2\Omega^2}{c^4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{1+\cos^2\theta}{r}\cos(2\Omega t - kr - 2\varphi) & \dfrac{2\cos\theta}{r}\sin(2\Omega t - kr - 2\varphi) & 0 \\ 0 & \dfrac{2\cos\theta}{r}\sin(2\Omega t - kr - 2\varphi) & \dfrac{1+\cos^2\theta}{r}\cos(2\Omega t - kr - 2\varphi) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note: using Kepler's third law $T^2 = 4\pi^2 a^3 / GM$ here to write $a = \left(GM/\Omega^2\right)^{1/3}$ would reveal that the gravitational wave amplitude depends on the orbital frequency of the binary as $\propto \Omega^{2/3}$.

What can we say about these? First, the **frequency** of the emitted gravitational waves is *twice* the orbital frequency of the binary, $\omega = 2\Omega$. This makes sense since the quadrupole moment repeats itself after every 180° (it just flips sign), so after just *half* an orbit for the binary, the gravitational waves emitted from it have already gone through *one* full period.



We can also see that the effect of the **azimuthal angle φ** of our observation point is only a phase shift in the wave, namely, a phase shift of 2φ. Both of these factors of 2 are reflections of the *tensor* (or *spin-2*) characteristics of gravitational waves.

Another interesting feature is that the amplitudes (ϵ_+ and $\epsilon_\times$) of the two polarization modes are not constant anymore in this case. Rather, they both depend on r and θ:

$$\epsilon_+ = \frac{2G\mu a^2\Omega^2}{c^4}\frac{1+\cos^2\theta}{r}$$
$$\epsilon_\times = \frac{4G\mu a^2\Omega^2}{c^4}\frac{\cos\theta}{r}$$

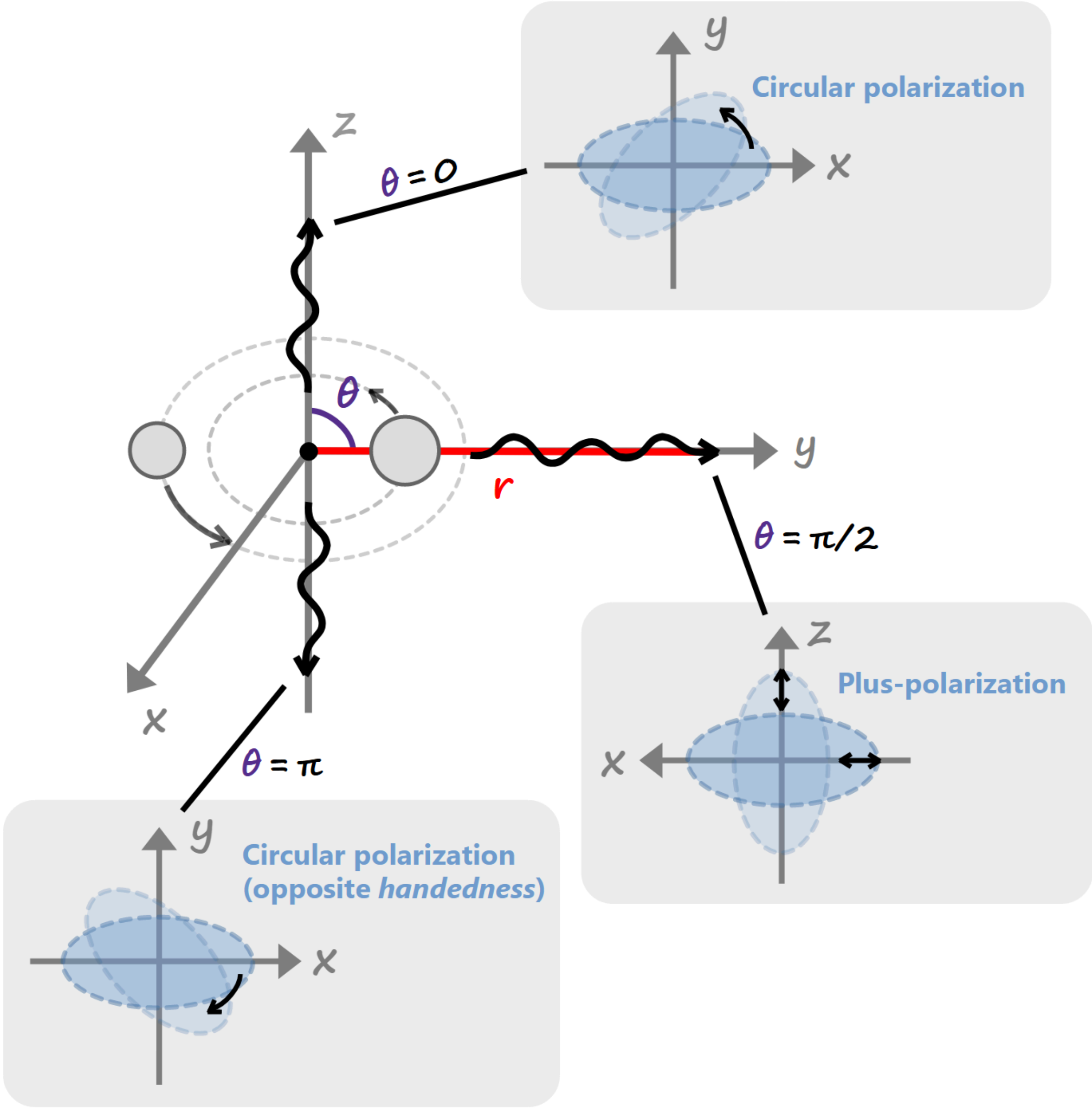
These fall off with distance to the source as $\sim r^{-1}$, which reflects the fact that the waves are spreading out from the source in *all* directions. The further they travel, the larger an area they will have spread out over, so the amplitude in any *one* direction will be smaller. So, "real" gravitational waves tend to get weaker the further they propagate from their source through this $\sim r^{-1}$-dependence - actually just like electromagnetic waves.

The most interesting feature might be the θ-dependence of the two polarization modes. The coordinate θ is the polar or inclination angle of our observation point, which means that the polarization ($+$ or $\times$) of the emitted waves we observe depends on the **inclination of the orbital plane** with respect to our observation point. When the emitted waves arrive at *our location*, they can appear either plus-polarized or some combination of plus- and cross-polarizations depending on our observation angle. They are never purely cross-polarized since $1+\cos^2\theta \neq 0$ (unless we were to rotate our coordinate system):

- At $\theta = \pi/2$ (when we're aligned with the orbital plane), $\epsilon_\times = 0$ but $\epsilon_+ \neq 0$: the emitted gravitational waves are **completely plus-polarized**.

- At $\theta = 0$ or $\theta = \pi$ (looking at the orbital plane "from above" or "below"), $\epsilon_+ = \epsilon_\times$: the waves consist of an equal amount of plus- and cross-polarizations, which is a **circularly polarized** gravitational wave. The polarization pattern is a *rotating ellipse*.

- At any other inclination, we'll generally find $\epsilon_+ \neq \epsilon_\times$ and that both polarization modes are non-zero. This would be an **elliptically polarized** gravitational wave.



z
y
x
r
θ
$\theta = 0$
$\theta = \pi/2$
$\theta = \pi$
Circular polarization
Plus-polarization
Circular polarization (opposite *handedness*)